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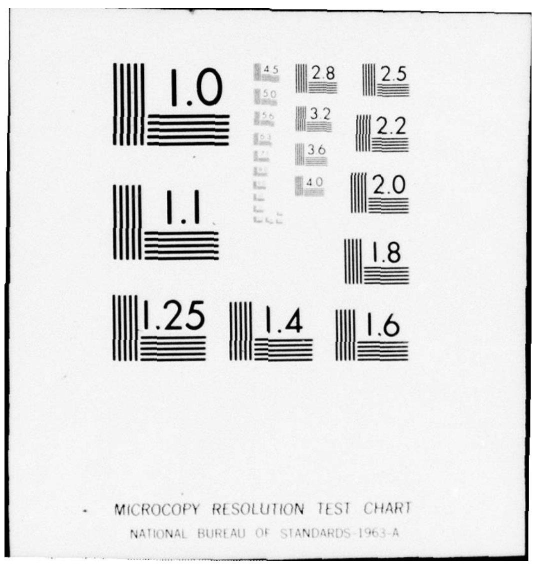
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Exact Equations of Motion for a Deformable Body

Engineering Science Operations
~~The~~ Aerospace Corporation
El Segundo, Calif. 90245

29 March 1977

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This report has been reviewed by the Information Office (OI) and is releasable to the National Technical Information Service (NTIS). At NTIS, it will be available to the general public, including foreign nations.

This technical report has been reviewed and is approved for publication. Publication of this report does not constitute Air Force approval of the report's findings or conclusions. It is published only for the exchange and stimulation of ideas.

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The exact equations of motion for an arbitrary deformable body undergoing large deformation are presented in two forms. One of the forms is a "momentum formulation" and the other is a "velocity formulation." Both forms of the equations of motion are derived by vectorial methods, but the resulting equations are shown to be the same as those obtained from a Hamiltonian or Lagrangian formulation. Essentially all the equations of motion in the spacecraft dynamics literature have the general form of the

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19. KEY WORDS (Continued)

20. ABSTRACT (Continued)

equations presented herein, even though the equations in the literature are obtained by various diverse methods (some vectorial and some variational). The generality of the final equations is due in part to modeling the deformable body as N particles and separating the degrees of freedom into 6 "external" or "rigid body" and into n $\leq$ 3N - 6 "internal" or "deformation" degrees of freedom. The internal or deformation degrees of freedom are represented by n generalized coordinates. There is no assumption that these internal or deformation coordinates are small or that their time derivatives are small.

The significance and novelty in the equations presented herein is that these equations put various alternative spacecraft dynamics formulations in an abstract perspective. This perspective makes it easier to compare various alternative approaches to the equations of motion. The "transformation operator formalism," developed previously by this author, is an essential ingredient in the derivation of the final equations presented herein, even though these equations could be obtained without the use of that formalism.

The final equations of motion are independent of the material properties of the body under consideration. In order to make specific use of these equations, specific constitutive equations must be postulated. The most typical constitutive equations used in the spacecraft dynamics literature is to consider the deformable body to consist of a collection of hinged rigid or linearly elastic sub-bodies; then the deformation within one of these sub-bodies is small, but the deformation between the sub-bodies may be arbitrarily large.

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PREFACE

This report is part of a continuing effort in the Controls Department of The Aerospace Corporation, of developing techniques for analyzing and simulating the dynamics and control of large flexible spacecraft. The following are previous related publications.

- Jerkovsky, W., "The Transformation Operator Approach to Multi-Body Spacecraft Dynamics. Volume I, The Transformation Operator Formulation; Part 1, Momentum Approach," Aerospace Corp., El Segundo, Calif., TOR-0075(5624-01)-1, 6 December 1974.

- Jerkovsky, W., "The Transformation Operator Approach to Multi-Body Dynamics," Aerospace Corp., El Segundo, Calif., TR-0076(6901-03)-5, 10 May 1976.

Additional related reports are in preparation.

The present report describes work performed under Engineering Methods of the Guidance and Control Division, Engineering Science Operations. I wish to thank Ms. Karen Saito for her skillful typing of the manuscript.

CONTENTS

I. INTRODUCTION

During the last twenty years, the control system analyst has been faced with modeling the dynamics of multibody spacecraft with ever increasing fidelity. This requirement for increased dynamics modeling sophistication is due in part to increasing control accuracy requirements, increasing spacecraft maneuvering requirements, increasing spacecraft size and structural flexibility, and finally, an increasing number of moving parts on board the spacecraft. A considerable number of papers[1-16] has been devoted to this subjected prior to 1970, and an even larger number[17-44] since 1970. The dynamics equations can get so complicated (because of the large number of terms) that it is difficult to see the forest for the trees. A similiar proliferation of terms in dynamics equations has occurred in the study of the dynamics of mechanisms and linkages[45-54].

The purpose of the present paper is to describe the general form of the dynamics equations of motion for an arbitrary deformable body undergoing large deformations. In order to keep the equations completely general, no particular material properties (i.e. no constitutive equations[55-56]) will be introduced . In this respect, the present paper is similiar to that of McDonough[57] except that we do not introduce any equations from continuum mechanics. Instead, we consider the deformable body to consist of a large number of particles (say 10^{25} molecules) of negligible inertia[58-61]. The deformable body consisting of N particles then has 6 "external" and n "internal" degrees of freedom, where $n \leq 3N - 6$; if there are no constraints among the N particles then n equals $3N - 6$, but if there are s constraints then $n = 3N - 6 - s$. The separation into "external" or "rigid body" degrees of freedom and "internal" or "deformation" degrees of

freedom is similiar to that introduced by Teixeira and Kane[62]. If the equations of motion are linearized in the deformations, then our equations reduce to the corresponding equations in the literature. If the deformations are taken to be zero, then we recover the equations of motion for a rigid body. A topological tree of rigid bodies can be handled by letting the deformation coordinates represent the relative gimbal angles (and relative hinge displacements, if any) between contiguous rigid bodies. Almost all the spacecraft dynamics equations in the literature can be obtained by introducing appropriate specializations into the equations presented herein. However, there are two types of spacecraft dynamics problems not considered herein: the deployment of roll-out booms or solar arrays, and mass expulsion as in rocket engine firing.

There are a number of controversies and misconceptions in the spacecraft dynamics literature on several basic mechanics issues. One of these controversies deals with whether a "Newton-Euler" (or "vectorial") dynamics formulation is preferable to a "Lagrange-Hamilton" (or "analytical dynamics" or "variational dynamics") formulation. Often it is claimed that a "Newton-Euler" formulation has the advantage of being "more physical", but a "Lagrange-Hamilton" formulation is preferable because constraint forces are automatically eliminated. We will show in this paper that our approach (which might be called "algebraical dynamics") has both of the above advantages. Our equations have the <u>form</u> of those of Newton and Euler, but the <u>procedure</u> we use to get the equations is essentially due to Lagrange and Hamilton. Our approach is similiar to that of Kane[63-65] except we make extensive use of transformation operators[66-69] to make velocity transformations, which in turn

induce appropriate momentum and force transformations. The transformation operator formalism is based on Kron's method of subspaces[70-73], which in turn is based on the procedure used by Lagrange to go from Newton's vector equations for particles to Lagrange's equations in terms of generalized coordinates.

Another controversy centers about whether it is preferable to use velocity or momentum as state variables. Russell has been the principal proponent of the momentum formulation of spacecraft dynamics problem. Recently, Vance and Sitchin[74-76] have also been advocating the momentum formulation. The controversy here is similiar to the question of whether Lagrange's or Hamilton's equations are preferable for spacecraft dynamics equations. We will see that our approach allows us to get the Lagrange and the Hamilton equations in a simple algebraic manner which does not require forming Lagrangians and Hamiltonians and then taking partial derivatives. Thus, our approach allows us to get both the velocity and the momentum equations for the deformable body.

Nonlinear dynamics equations for a deformable body can be obtained by expressing the particle displacements (relative to a fixed or floating reference frame) in terms of the eigenfunctions or eigenvectors of the corresponding (or closely related) linearized equations of motion. Different choices of reference frames[58,5,77-79] and different choice of eigenfunctions or eigenvectors can be made. We will maintain generality by not specifying the form or nature of the particle displacement field; we merely assume that some displacement field function does exist, and we assume this function is continuous and has continuous partial derivatives of up to the second order (so that we can interchange the order of two partial differentiations).

The dynamics equations for a spacecraft are sometimes written relative to a point fixed in the structure, and sometimes they are written relative to the total center of mass. We will maintain generality by writing them relative to an arbitrary point which may be fixed in the structure or not; in particular, the point may be the total center of mass.

Our dynamics equations for a deformable body are given in Section V, page 65. Our "momentum formulation" equations are $\dot{\overline{G}} + \overline{X} = \overline{K}$ and $\overline{G} = \overline{\mu} \cdot \overline{\sigma}$, where $\overline{\sigma}$ is the system velocity, $\overline{G}$ is the system momentum (which is defined so that the kinetic energy is given by $T = \frac{1}{2} \overline{G}^t \cdot \overline{\sigma}$), $\overline{\mu}$ is the system mass (which yields $T = \frac{1}{2} \overline{\sigma}^t \cdot \overline{\mu} \cdot \overline{\sigma}$), $\overline{K}$ is the system force (which yields $\dot{T} = \overline{K}^t \cdot \overline{\sigma}$) and $\overline{X}$ is an extra term which must be added to the time derivative of $\overline{G}$ so that the result is $\overline{K}$. Our "velocity formulation" equation is $\overline{\mu} \cdot \dot{\overline{\sigma}} + \overline{Y} = \overline{K}$ where $\overline{Y}$ is an extra term which must be added to the system mass times the system acceleration so that the result is $\overline{K}$. Of course, our dynamics equations are not adequately described until each of the variables in the equations is well defined. The variables for the "external" or "rigid body" degrees of freedom are described in Section II in terms of the corresponding variables for the individual particles which constitute the deformable body. Section III describes the variables for the "internal" or "deformation" degrees of freedom. Most of Section IV is taken up in determining first the general forms, and then the explicit expressions for $\overline{X}$ and $\overline{Y}$; the expressions for $\overline{\mu}$ and $\overline{K}$ are actually quite straightforward.

II. CLASSICAL VECTOR EQUATIONS

We will start out with Newton's law for a particle, written relative to an inertial origin. This law applies to each of the particles of the system of particles which represents our deformable body. By assuming that action is equal and opposite to reaction, the internal forces drop out of the equations which describe the external degrees of freedom; however, these internal forces do enter the equations for the internal degrees of freedom (except that purely constraint forces drop out).

A. EQUATIONS RELATIVE TO AN INERTIAL ORIGIN

In this subsection, we introduce the vector notation and equations[80-81] which we use to describe the deformable body relative to an inertial origin. We number the particles from 1 to N and we refer to a typical particle as the i^{th} particle or particle i, where $1 \le i \le N$. Let $\vec{r}_i$ be the position vector of the i^{th} particle relative to an inertial reference origin. $\vec{r}_i$ will, in general, vary with time, t. The velocity of the i^{th} particle, $\vec{v}_i$, is equal to the inertial time derivative of $\vec{r}_i$, and is denoted by

$$\vec{v}_i = \dot{\vec{r}}_i = \frac{d\vec{r}_i}{dt} \tag{2-1}$$

This is a "material" time derivative[55-56] since it is the velocity experienced by a particular material point of the deformable body.

Let m_i be the mass of the i^{th} particle and let M be the total mass of all particles. Then a point c, the total center of mass, is given by the position vector $\vec{r}_c$ where

$$\vec{r}_c = \frac{1}{M} \sum_i m_i \vec{r}_i \qquad (2\text{-}2)$$

where

$$M = \sum_i m_i \qquad (2\text{-}3)$$

In Equations (2) and (3)* it is understood that the summation is over all the particles; i.e. for i = 1 to N. The velocity of the center of mass, $\vec{v}_c$, is given by

$$\vec{v}_c = \dot{\vec{r}}_c = \frac{1}{M} \sum_i m_i \vec{v}_i \qquad (2\text{-}4)$$

The linear momentum of particle i, $\vec{p}_i$, and of the whole system, $\vec{P}$, are given by

$$\vec{p}_i = m_i \vec{v}_i \qquad (2\text{-}5)$$

$$\vec{P} = \sum_i \vec{p}_i \qquad (2\text{-}6)$$

Combining Equations (4) to (6) yields

$$\vec{P} = \sum_i m_i \vec{v}_i = M \vec{v}_c \qquad (2\text{-}7)$$

The kinetic energy of particle i, T_i, and of the whole system, T, are given by

---

*We use the following convention: if a referenced equation appears in the same section as the reference, then the section number is deleted from the equation number.

$$T_i = \frac{1}{2} m_i \vec{v}_i^{\,2} = \frac{1}{2} \vec{p}_i \cdot \vec{v}_i = \frac{1}{2m_i} \vec{p}_i^{\,2} \tag{2-8}$$

$$T = \sum_i T_i \tag{2-9}$$

where $\vec{v}_i^{\,2} = \vec{v}_i \cdot \vec{v}_i$ and $\vec{p}_i^{\,2} = \vec{p}_i \cdot \vec{p}_i$. From Equations (8) and (5) we note that

$$\frac{\partial T_i}{\partial \vec{v}_i} = m_i \vec{v}_i = \vec{p}_i \tag{2-10}$$

$$\frac{\partial T_i}{\partial \vec{p}_i} = \frac{1}{m_i} \vec{p}_i = \vec{v}_i \tag{2-11}$$

It should be noted that in Equation (10) T_i is considered to be a function of $\vec{v}_i$ whereas in Equation (11) T_i is considered to be a function of $\vec{p}_i$. The confusion between these two different functions can be eliminated by using different symbols for the two kinetic energy functions, but for simplicity we will not do so. Since each of the particles is independent, Equations (8) to (10) yield

$$\frac{\partial T}{\partial \vec{v}_i} = \frac{\partial T_i}{\partial \vec{v}_i} = \vec{p}_i \tag{2-12}$$

$$\frac{\partial T}{\partial \vec{p}_i} = \frac{\partial T_i}{\partial \vec{p}_i} = \vec{v}_i \tag{2-13}$$

Thus, from the system kinetic energy function, T, the i^{th} particle momentum and the i^{th} particle velocity can be obtained by partial differentiation.

Let $\vec{f}_i$ be the force on the i^{th} particle. Then Newton's law of motion can be written as

$$\dot{\vec{p}}_i = \vec{f}_i \tag{2-14}$$

or

$$m_i \dot{\vec{v}}_i = \vec{f}_i \tag{2-15}$$

or

$$m_i \ddot{\vec{r}}_i = \vec{f}_i \tag{2-16}$$

Equation (14) is a "momentum equation" and Equation (15) is a "velocity equation." Since the mass m_i is a constant, the momentum and velocity equations differ only by this constant factor. Let $\vec{F}$ denote the total force on the system:

$$\vec{F} = \sum_i \vec{f}_i \tag{2-17}$$

Now we can write $\vec{f}_i$ as the sum of $\vec{f}_i^{\,E}$ which is an external force (whose origin is from outside of the deformable body) and $\vec{f}_i^{\,I}$ which is an internal force (due to the coupling among the mass points of the body). We assume Newton's third law, that action is equal and opposite to reaction, and conclude that

$$\sum_i \vec{f}_i^{\,I} = \vec{0} \tag{2-18}$$

Hence

$$\vec{F} = \sum_i \left(\vec{f}_i^{\,E} + f_i^{\,I} \right) = \sum_i \vec{f}_i^{\,E} = \vec{F}^E \tag{2-19}$$

Combining Equations (6), (14), and (19) now yields

$$\dot{\vec{P}} = \vec{F} = \vec{F}^E \tag{2-20}$$

Thus, Newton's law for a particle also holds for the composite system linear momentum.

Taking the time derivative of the kinetic energy of the i^{th} particle yields

$$\dot{T}_i = m\vec{v}_i \cdot \dot{\vec{v}}_i = \vec{f}_i \cdot \vec{v}_i \tag{2-21}$$

The time derivative of the kinetic energy for the entire system is

$$\dot{T} = \sum_i \dot{T}_i = \sum_i \vec{f}_i \cdot \vec{v}_i \tag{2-22}$$

Note that the time derivative of the total kinetic energy depends on both internal and external forces; i.e. the total kinetic energy can change with time even in the absence of external forces.

We have now written all the fundamental equations relative to an inertial origin. In the next subsection we write these equations relative to a moving (possibly accelerating) origin.

B. EQUATIONS RELATIVE TO A MOVING ORIGIN

We now introduce an arbitrary point, a, with position vector $\vec{r}_a$ and with velocity $\vec{v}_a = \dot{\vec{r}}_a$. Let $\vec{R}_{ia}$ be the position vector to particle i (or point i) from point a. From Figure 1, it is evident that we can now write

$$\vec{r}_i = \vec{r}_a + \vec{R}_{ia} \tag{2-23}$$

Multiplying this by m_i and summing overall i yields

$$M\vec{r}_c = \sum_i m_i \vec{r}_i = M\vec{r}_a + M\vec{R}_{ca} \quad (\text{or } \vec{r}_c = \vec{r}_a + \vec{R}_{ca}) \tag{2-24}$$

where

$$M\vec{R}_{ca} = \sum_i m_i \vec{R}_{ia} \tag{2-25}$$

Thus, $\vec{R}_{ca}$ is the position vector to the center of mass, c, from the point a. Taking the inertial time derivative of Equation (23) yields

$$\vec{v}_i = \vec{v}_a + \dot{\vec{R}}_{ia} \tag{2-26}$$

The arbitrary point a may coincide with (be equal to) c. In that case, Equations (23) and (26) become

$$\vec{r}_i = \vec{r}_c + \vec{R}_{ic} \tag{2-27}$$

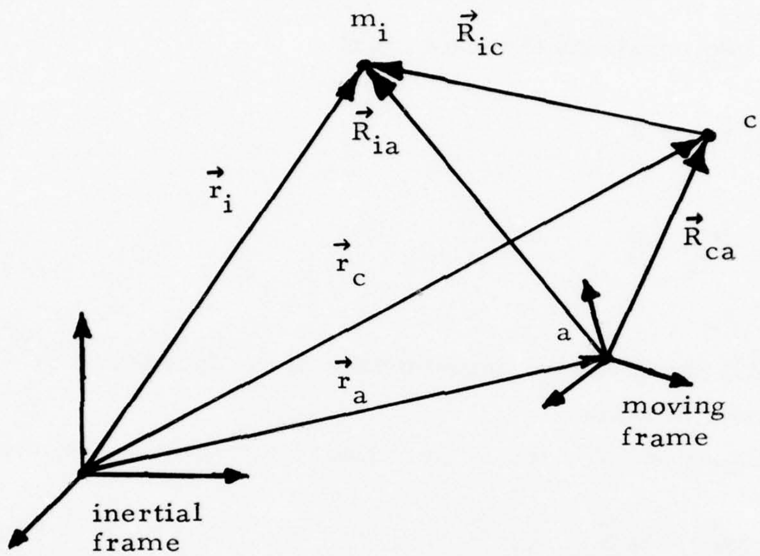

Figure 1. Diagram of Position Vectors For a System of Particles

$$\vec{v}_i = \vec{v}_c + \dot{\vec{R}}_{ic} \tag{2-28}$$

From these two equations it follows that

$$\sum_i m_i \vec{R}_{ic} = \vec{0} \tag{2-29}$$

$$\sum_i m_i \dot{\vec{R}}_{ic} = \vec{0} \tag{2-30}$$

Equations (29) and (30) are consequences of the fact that c is the total center of mass.

Using Equation (26), the system linear momentum becomes

$$\vec{P} = M\vec{v}_a + M\dot{\vec{R}}_{ca} \tag{2-31}$$

where, from Equation (25),

$$M\dot{\vec{R}}_{ca} = \sum_i m_i \dot{\vec{R}}_{ia} \tag{2-32}$$

If a coincides with c, then Equation (31) reduces to $\vec{P} = M\vec{v}_c$ because $\dot{\vec{R}}_{cc} = 0$.

The system kinetic energy can now be written as

$$T = \frac{1}{2} M\vec{v}_a^{\,2} + M\vec{v}_a \cdot \dot{\vec{R}}_{ca} + \frac{1}{2} \sum_i m_i \dot{\vec{R}}_{ia}^{\,2} \tag{2-33}$$

Note that the partial derivative of T with respect to $\vec{v}_a$ is $\vec{P}$:

$$\frac{\partial T}{\partial \vec{v}_a} = M\vec{v}_a + M\dot{\vec{R}}_{ca} = \vec{P} \tag{2-34}$$

If a coincides with c, then Equations (33) and (34) reduce to

$$T = \frac{1}{2} M\vec{v}_c^{\,2} + \frac{1}{2} \sum_i m_i \dot{\vec{R}}_{ic}^{\,2} \tag{2-35}$$

$$\frac{\partial T}{\partial \vec{v}_c} = M\vec{v}_c = \vec{P} \tag{2-36}$$

As is usual when taking partial derivatives, Equations (34) and (36) have to be interpreted carefully because the two equations deal with different functions. If T_a^* denotes the kinetic energy function in terms of the independent variable $\vec{v}_a$, and T_c^* denotes this function in terms of $\vec{v}_c$, then we can write

$$\vec{P} = \frac{\partial T_a^*}{\partial \vec{v}_a} = \frac{\partial T_c^*}{\partial \vec{v}_c}$$

From Equations (22) and (26), we get for the time derivative of the system kinetic energy

$$\dot{T} = \sum_i \vec{f}_i \cdot \vec{v}_i = \vec{F} \cdot \vec{v}_a + \sum_i \vec{f}_i \cdot \dot{\vec{R}}_{ia} \tag{2-37}$$

where $\vec{F} = \vec{F}^E$ is the total external force.

The angular momentum about the point a of particle i, $\vec{h}_{ai}$, and of the whole system, $\vec{H}_a$, are given by

$$\vec{h}_{ai} = \vec{R}_{ia} \times \vec{p}_i \tag{2-38}$$

$$\vec{H}_a = \sum_i \vec{h}_{ai} \tag{2-39}$$

Thus, $\vec{H}_a$ is given by

$$\vec{H}_a = \sum_i \vec{R}_{ia} \times \vec{p}_i = \sum_i m_i \vec{R}_{ia} \times \vec{v}_i \tag{2-40}$$

If a coincides with c, this expression becomes

$$\vec{H}_c = \sum_i \vec{R}_{ic} \times \vec{p}_i = \sum_i m_i \vec{R}_{ic} \times \vec{v}_i \tag{2-41}$$

If we write $\vec{R}_{ia} = \vec{R}_{ic} + \vec{R}_{ca}$ in Equation (40), we find that

$$\vec{H}_a = \vec{H}_c + \vec{R}_{ca} \times \vec{P} \tag{2-42}$$

We get an alternate expression for $\vec{H}_a$ by expressing $\vec{v}_i$ in terms of $\vec{v}_a$ as given by Equation (26). Then

$$\vec{H}_a = M\vec{R}_{ca} \times \vec{v}_a + \sum_i m_i \vec{R}_{ia} \times \dot{\vec{R}}_{ia} \tag{2-43}$$

If a coincides with c, we get an alternate expression for $\vec{H}_c$:

$$\vec{H}_c = \sum_i m_i \vec{R}_{ic} \times \dot{\vec{R}}_{ic} \tag{2-44}$$

Taking the time derivative of Equation (40) yields

$$\dot{\vec{H}}_a = \sum_i \dot{\vec{R}}_{ia} \times \vec{p}_i + \sum_i \vec{R}_{ia} \times \dot{\vec{p}}_i \qquad (2\text{-}45)$$

We substitute $\dot{\vec{p}}_i = \vec{f}_i$ on the right hand side, and then introduce the torque or moment of force, $\vec{L}_a$, on the system about the point a:

$$\vec{L}_a = \sum_i \vec{R}_{ia} \times \vec{f}_i = \sum_i \vec{R}_{ia} \times \vec{f}_i^{\,E} = \vec{L}_a^{\,E} \qquad (2\text{-}46)$$

where we have assumed that action is equal and opposite to reaction and hence

$$\sum_i \vec{R}_{ia} \times \vec{f}_i^{\,I} = \vec{0} \qquad (2\text{-}47)$$

Next we note that

$$\sum_i \dot{\vec{R}}_{ia} \times \vec{p}_i = \sum_i (\dot{\vec{r}}_i - \dot{\vec{r}}_a) \times \vec{p}_i = -\vec{v}_a \times \vec{P} \qquad (2\text{-}48)$$

Combining Equations (45) and (48) and making use of Equation (47) now yields

$$\dot{\vec{H}}_a + \vec{v}_a \times \vec{P} = \vec{L}_a \qquad (2\text{-}49)$$

If a coincides with c, this reduces to

$$\dot{\vec{H}}_c = \vec{L}_c \qquad (2\text{-}50)$$

since $\vec{v}_c \times \vec{P} = \vec{0}$.

The definition of the total torque about the point a, Equation (46), and the definition of the total angular momentum about the point a, Equation (40), are very similiar. This similiarity can be enhanced by introducing the moment of force about the point a on particle i, $\vec{\ell}_{ai}$, as follows

$$\vec{\ell}_{ai} = \vec{R}_{ia} \times \vec{f}_i \tag{2-51}$$

Then

$$\vec{L}_a = \sum_i \vec{\ell}_{ai} \tag{2-52}$$

Note the similiarity of Equations (51) and (52) to Equations (38) and (39), respectively.

The angular momentum is a fundamental quantity but we cannot obtain it by partial differentiation of the kinetic energy function given in Equation (33). This situation will be remedied in the next subsection where we introduce an angular velocity, and then the angular momentum can be obtained by partial differentiation of the kinetic energy with respect to this angular velocity.

C. EQUATIONS RELATIVE TO A ROTATING FRAME

In the last subsection, we obtained the following expressions for the linear momentum, $\vec{P}$, angular momentum about a, $\vec{H}_a$, kinetic energy, T, and time derivative of kinetic energy, $\dot{T}$:

$$\vec{P} = M\vec{v}_a + \sum_i m_i \dot{\vec{R}}_{ia} \tag{2-53}$$

$$\vec{H}_a = M\vec{R}_{ca} \times \vec{v}_a + \sum_i m_i \vec{R}_{ia} \times \dot{\vec{R}}_{ia} \tag{2-54}$$

$$T = \frac{1}{2} M\vec{v}_a^{\,2} + \sum_i m_i \dot{\vec{R}}_{ia} \cdot \vec{v}_a + \frac{1}{2} \sum_i m_i \dot{\vec{R}}_{ia}^{\,2} \tag{2-55}$$

$$\dot{T} = \vec{F} \cdot \vec{v}_a + \sum_i \vec{f}_i \cdot \dot{\vec{R}}_{ia} \tag{2-56}$$

We will now replace the inertial time derivative of $\vec{R}_{ia}$ in these equations by the time derivative with respect to a frame (which we can call "frame B") which has angular velocity $\vec{\omega}^B$ with respect to inertial space. We know that for any vector $\vec{V}$ we have

$$\dot{\vec{V}} = \overset{B}{\vec{V}} + \vec{\omega}^B \times \vec{V} \tag{2-57}$$

where $\overset{B}{\vec{V}}$ denotes the time derivative of $\vec{V}$ with respect to the frame with angular velocity $\vec{\omega}^B$(i.e. with respect to frame B). Applied to the vector $\vec{R}_{ia}$, Equation (57) yields

$$\dot{\vec{R}}_{ia} = \overset{B}{\vec{R}}_{ia} + \vec{\omega}^B \times \vec{R}_{ia} \tag{2-58}$$

It is convenient to let $\vec{u}^B_{ia}$ denote $\overset{B}{\vec{R}}_{ia}$:

$$\vec{u}^B_{ia} = \overset{B}{\vec{R}}_{ia} \tag{2-59}$$

It is also convenient to write

$$\vec{\omega}^B \times \vec{R}_{ia} = \tilde{R}^t_{ia} \cdot \vec{\omega}^B \tag{2-60}$$

where* $\widetilde{R}_{ia}$ is the *skew-symmetric* dyadic formed from the vector $\vec{R}_{ia}$, and $\widetilde{R}^t_{ia}$ is the dyadic transpose (or conjugate) of $\widetilde{R}_{ia}$. Thus, $\dot{\vec{R}}_{ia}$ can be expressed as

$$\dot{\vec{R}}_{ia} = \vec{u}^B_{ia} + \widetilde{R}^t_{ia} \cdot \vec{\omega}^B \tag{2-61}$$

Substituting Equation (61) into Equations (53) to (56) yields

$$\vec{P} = M\vec{v}_a + M\widetilde{R}^t_{ca} \cdot \vec{\omega}^B + \sum_i m_i \vec{u}^B_{ia} \tag{2-62}$$

$$\vec{H}_a = M\widetilde{R}_{ca} \cdot \vec{v}_a + \overleftrightarrow{I}_a \cdot \vec{\omega}^B + \sum_i m_i \widetilde{R}_{ia} \cdot \vec{u}^B_{ia} \tag{2-63}$$

$$T = \frac{1}{2} M\vec{v}_a^{\,2} + \sum_i m_i \vec{u}^B_{ia} \cdot \vec{v}_a + \vec{v}_a \cdot M\widetilde{R}^t_{ca} \cdot \vec{\omega}^B + \frac{1}{2}\vec{\omega}^B \cdot \overleftrightarrow{I}_a \cdot \vec{\omega}^B + \sum_i m_i \vec{u}^B_{ia} \cdot \widetilde{R}^t_{ia} \cdot \vec{\omega}^B + \frac{1}{2}\sum_i m_i \vec{u}^{B\,2}_{ia} \tag{2-64}$$

$$\dot{T} = \vec{F} \cdot \vec{v}_a + \vec{L}_a \cdot \vec{\omega}^B + \sum_i \vec{f}_i \cdot \vec{u}^B_{ia} \tag{2-65}$$

where

$$\overleftrightarrow{I}_a = \sum_i m_i \widetilde{R}_{ia} \cdot \widetilde{R}^t_{ia} = \overleftrightarrow{I}^t_a \tag{2-66}$$

Evidently, $\overleftrightarrow{I}_a$ is the inertia[81] of the deformable body about the arbitrary point a. If a coincides with c, we have

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* See Appendix A

$$\overleftrightarrow{I}_c = \sum_i m_i \widetilde{R}_{ic} \cdot \widetilde{R}^t_{ic} \tag{2-67}$$

If we write $\vec{R}_{ia} = \vec{R}_{ic} + \vec{R}_{ca}$ in Equation (66), we find that

$$\overleftrightarrow{I}_a = \overleftrightarrow{I}_c + M\widetilde{R}_{ca} \cdot \widetilde{R}^t_{ca} \tag{2-68}$$

Considering $\vec{v}_a$ and $\vec{\omega}^B$ to be independent variables in the kinetic energy function given by Equation (64), we see that

$$\frac{\partial T}{\partial \vec{v}_a} = \vec{P} \tag{2-69}$$

$$\frac{\partial T}{\partial \vec{\omega}^B} = \vec{H}_a \tag{2-70}$$

Here the partial derivatives of T are understood to mean the partial derivatives of the function T_a^{B*} given in Equation (64). The time derivatives of $\vec{P}$ and $\vec{H}_a$ are still given by

$$\dot{\vec{P}} = \vec{F} \tag{2-71}$$

$$\dot{\vec{H}}_a + \vec{v}_a \times \vec{P} = \vec{L}_a \tag{2-72}$$

Equations (69) to (72) are very simple but very general equations which hold for any deformable body, even a body containing fluids[82]. However, Equations (69) to (72) describe only the 6 external degrees of freedom of the deformable body. In order to describe the $n \leq 3N - 6$ internal degrees of freedom of the system

of N particles, we must somehow introduce n appropriate coordinates and velocities. In addition, it would be desirable to introduce n appropriate momenta.

It is instructive to write Equations (62) and (63) in the following matrix form*

$$\begin{bmatrix} \vec{P} \\ \\ \vec{H}_a \end{bmatrix} = \begin{bmatrix} M\overleftrightarrow{E} & M\widetilde{R}^t_{ca} & m_1\overleftrightarrow{E} & m_2\overleftrightarrow{E} & \cdots & m_N\overleftrightarrow{E} \\ \\ M\widetilde{R}_{ca} & \overleftrightarrow{I}_a & m_1\widetilde{R}_{1a} & m_2\widetilde{R}_{2a} & \cdots & m_N\widetilde{R}_{Na} \end{bmatrix} \cdot \begin{bmatrix} \vec{v}_a \\ \vec{\omega}^B \\ \vec{u}^B_{1a} \\ \vec{u}^B_{2a} \\ \cdot \\ \cdot \\ \cdot \\ \vec{u}^B_{Na} \end{bmatrix} \qquad (2\text{-}73)$$

where $\overleftrightarrow{E}$ is the identity dyadic. Since the system has only n internal degrees of freedom, the relative velocity vectors $\vec{u}^B_{1a}, \vec{u}^B_{2a}, \cdots, \vec{u}^B_{Na}$ must be expressed in terms of only n coordinates and velocities. Also, there are n more momenta which characterize the internal degrees of freedom of the system.

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*See Appendix B.

III. INTERNAL GENERALIZED COORDINATES

If there are n internal degrees of freedom described by internal generalized coordinates $\xi^B_{a1}, \xi^B_{a2}, \cdots, \xi^B_{an}$, then each $\vec{u}^B_{ia}$ must be a linear combination of the time derivatives of these internal generalized coordinates. Thus, we can write an equation of the form

$$\vec{u}^B_{ia} = \sum_{j=1}^{n} \vec{\Phi}^B_{iaj} \dot{\xi}^B_{aj} \tag{3-1}$$

where $\vec{\Phi}^B_{iaj}$ in general, depends on the internal generalized coordinates $\xi^B_{a1}, \xi^B_{a2}, \cdots, \xi^B_{an}$; however, $\vec{\Phi}^B_{ia}$ does not depend on the time derivatives of the internal generalized coordinates.

Now $\vec{R}_{ia}$ is a function of the internal generalized coordinates, and we assume that $\vec{R}_{ia}$ is not an explicit function of time. Hence

$$\vec{u}^B_{ia} = \overset{B}{\dot{\vec{R}}}_{ia} = \sum_j \frac{\partial \vec{R}_{ia}}{\partial \xi^B_{aj}} \dot{\xi}^B_{aj} \tag{3-2}$$

Comparing Equations (1) and (2), we see that

$$\vec{\Phi}^B_{iaj} = \frac{\partial \vec{R}_{ia}}{\partial \xi^B_{aj}} \tag{3-3}$$

Equation (1) can be written conveniently in matrix form as follows*:

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* See Appendix B

$$\vec{u}^{B}_{ia} = \begin{bmatrix} \vec{\Phi}^{B}_{ia1} & \vec{\Phi}^{B}_{ia2} & \cdots & \vec{\Phi}^{B}_{ian} \end{bmatrix} \begin{bmatrix} \dot{\xi}^{B}_{a1} \\ \dot{\xi}^{B}_{a2} \\ \cdot \\ \cdot \\ \cdot \\ \dot{\xi}^{B}_{an} \end{bmatrix} = \begin{bmatrix} \dot{\xi}^{B}_{a1} & \dot{\xi}^{B}_{a2} & \cdots & \dot{\xi}^{B}_{an} \end{bmatrix} \begin{bmatrix} \vec{\Phi}^{B}_{ia1} \\ \vec{\Phi}^{B}_{ia2} \\ \cdot \\ \cdot \\ \cdot \\ \vec{\Phi}^{B}_{ian} \end{bmatrix} \quad (3\text{-}4)$$

This can be written more compactly by introducing the following notation

$$\Phi^{B}_{ia} = \begin{bmatrix} \vec{\Phi}^{B}_{ia1} \\ \vec{\Phi}^{B}_{ia2} \\ \cdot \\ \cdot \\ \cdot \\ \vec{\Phi}^{B}_{ian} \end{bmatrix} \quad (3\text{-}5)$$

$$\Phi^{B^{t}}_{ia} = [\, \vec{\Phi}^{B}_{ia1} \quad \vec{\Phi}^{B}_{ia2} \quad \cdots \quad \vec{\Phi}^{B}_{ian} \,] \quad (3\text{-}6)$$

$$\dot{\xi}^{B}_{a} = \begin{bmatrix} \dot{\xi}^{B}_{a1} \\ \dot{\xi}^{B}_{a2} \\ \cdot \\ \cdot \\ \cdot \\ \dot{\xi}^{B}_{an} \end{bmatrix} \quad (3\text{-}7)$$

$$\dot{\xi}_a^{B^t} = \left[\dot{\xi}_{a1}^B \quad \dot{\xi}_{a2}^B \quad \cdots \quad \dot{\xi}_{an}^B \right] \tag{3-8}$$

Equation (1) can now be written as

$$\vec{u}_{ia}^B = \Phi_{ia}^{B^t} \dot{\xi}_a^B = \dot{\xi}_a^{B^t} \Phi_{ia}^B \tag{3-9}$$

These two different forms of expressing $\vec{u}_{ia}^B$ are useful in forming quantities like $\vec{u}_{ia}^{B^2}$:

$$\vec{u}_{ia}^{B^2} = \vec{u}_{ia}^B \cdot \vec{u}_{ia}^B = \dot{\xi}_a^{B^t} \Phi_{ia}^B \cdot \Phi_{ia}^{B^t} \dot{\xi}_a^B \tag{3-10}$$

where $\vec{\Phi}_{ia}^B \cdot \vec{\Phi}_{ia}^{B^t}$ is an n × n matrix of scalars:

$$\Phi_{ia}^B \cdot \Phi_{ia}^{B^t} = \begin{bmatrix} \vec{\Phi}_{ia1}^B \\ \vec{\Phi}_{ia2}^B \\ \vdots \\ \vec{\Phi}_{ian}^B \end{bmatrix} \bullet \left[\vec{\Phi}_{ia1}^B \quad \vec{\Phi}_{ia2}^B \quad \cdots \quad \vec{\Phi}_{ian}^B \right] \tag{3-11}$$

$$= \begin{bmatrix} \vec{\Phi}_{ia1}^B \cdot \vec{\Phi}_{ia1}^B & \vec{\Phi}_{ia1}^B \cdot \vec{\Phi}_{ia2}^B & \cdots & \vec{\Phi}_{ia1}^B \cdot \vec{\Phi}_{ian}^B \\ \vec{\Phi}_{ia2}^B \cdot \vec{\Phi}_{ia1}^B & \vec{\Phi}_{ia2}^B \cdot \vec{\Phi}_{ia2}^B & \cdots & \vec{\Phi}_{ia2}^B \cdot \vec{\Phi}_{ian}^B \\ \vdots & \vdots & & \vdots \\ \vec{\Phi}_{ian}^B \cdot \vec{\Phi}_{ia1}^B & \vec{\Phi}_{ian}^B \cdot \vec{\Phi}_{ia2}^B & \cdots & \vec{\Phi}_{ian}^B \cdot \vec{\Phi}_{ian}^B \end{bmatrix}$$

Thus, the element in the j^{th} row and k^{th} column of $\Phi^B_{ia} \cdot \vec{\Phi}^{B^t}_{ia}$ is

$$\left(\Phi^B_{ia} \cdot \Phi^{B^t}_{ia}\right)_{jk} = \vec{\Phi}^B_{iaj} \cdot \vec{\Phi}^B_{iak} \tag{3-12}$$

Equations (9) and (10)can now be used to write Equations (2-62) to (2-65) in the following form

$$\vec{P} = M\vec{v}_a + M\tilde{R}^t_{ca} \cdot \vec{\omega}^B + \sum_i m_i \Phi^{B^t}_{ia} \dot{\xi}^B_a \tag{3-13}$$

$$\vec{H}_a = M\tilde{R}_{ca} \cdot \vec{v}_a + \overleftrightarrow{I}_a \cdot \vec{\omega}^B + \sum_i m_i \tilde{R}_{ia} \cdot \Phi^{B^t}_{ia} \dot{\xi}^B_a \tag{3-14}$$

$$\begin{aligned} T = {} & \frac{1}{2} M\vec{v}_a^{\,2} + \vec{v}_a \cdot M\tilde{R}^t_{ca} \cdot \vec{\omega}^B + \vec{v}_a \cdot \sum_i m_i \Phi^{B^t}_{ia} \dot{\xi}^B_a \\ & + \frac{1}{2} \vec{\omega}^B \cdot \overleftrightarrow{I}_a \cdot \vec{\omega}^B + \vec{\omega}^B \cdot \sum_i m_i \tilde{R}_{ia} \cdot \Phi^{B^t}_{ia} \dot{\xi}^B_a \\ & + \frac{1}{2} \dot{\xi}^{B^t}_a \sum_i m_i \Phi^B_{ia} \cdot \Phi^{B^t}_{ia} \dot{\xi}^B_a \end{aligned} \tag{3-15}$$

$$\dot{T} = \vec{F} \cdot \vec{v}_a + \vec{L}_a \cdot \vec{\omega}^B + \sum_i \vec{f}_i \cdot \Phi^{B^t}_{ia} \dot{\xi}^B_a \tag{3-16}$$

Note that in each of these equations we have a summation over all the particles (from i = 1 to N). We can eliminate this explicit dependence on the individual particles by introducing the following notation*

$$\alpha^B_a = \sum_i m_i \Phi^B_{ia} \tag{3-17}$$

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* See Appendix B

$$\alpha_a^{B^t} = \sum_i m_i \Phi_{ia}^{B^t} \tag{3-18}$$

$$\beta_a^B = \sum_i m_i \Phi_{ia}^B \cdot \tilde{R}_{ia}^t = \sum_i m_i \tilde{R}_{ia} \cdot \Phi_{ia}^B \tag{3-19}$$

$$\beta_a^{B^t} = \sum_i m_i \tilde{R}_{ia} \cdot \Phi_{ia}^{B^t} = \sum_i m_i \Phi_{ia}^{B^t} \cdot \tilde{R}_{ia}^t \tag{3-20}$$

$$\gamma_a^B = \sum_i m_i \Phi_{ia}^B \cdot \Phi_{ia}^{B^t} = \gamma_a^{B^t} \tag{3-21}$$

$$k_a^B = \sum_i \Phi_{ia}^B \cdot \vec{f}_i = \sum_i \vec{f}_i \cdot \Phi_{ia}^B \tag{3-22}$$

$$k_a^{B^t} = \sum_i \vec{f}_i \cdot \Phi_{ia}^{B^t} = \sum_i \Phi_{ia}^{B^t} \cdot \vec{f}_i \tag{3-23}$$

Equations (13)to (16) now become

$$\vec{P} = M\vec{v}_a + M\tilde{R}_{ca}^t \cdot \vec{\omega}^B + \alpha_a^{B^t} \dot{\xi}_a^B \tag{3-24}$$

$$\vec{H}_a = M\tilde{R}_{ca} \cdot \vec{v}_a + \overleftrightarrow{I}_a \cdot \vec{\omega}^B + \beta_a^{B^t} \dot{\xi}_a^B \tag{3-25}$$

$$T = \frac{1}{2} M\vec{v}_a^{\,2} + \vec{v}_a \cdot M\tilde{R}_{ca}^t \cdot \vec{\omega}^B + \vec{v}_a \cdot \alpha_a^{B^t} \dot{\xi}_a^B + \frac{1}{2} \vec{\omega}^B \cdot \overleftrightarrow{I}_a \cdot \vec{\omega}^B + \vec{\omega}^B \cdot \beta_a^{B^t} \dot{\xi}_a^B + \frac{1}{2} \dot{\xi}_a^{B^t} \gamma_a^B \dot{\xi}_a^B \tag{3-26}$$

$$\dot{T} = \vec{F} \cdot \vec{v}_a + \vec{L}_a \cdot \vec{\omega}^B + k_a^{B^t} \dot{\xi}_a^B \tag{3-27}$$

These equations can be written conveniently in matrix form as follows*:

$$\begin{bmatrix} \vec{P} \\ \\ \vec{H}_a \end{bmatrix} = \begin{bmatrix} M\overleftrightarrow{E} & M\tilde{R}^t_{ca} & \alpha^{B^t}_a \\ \\ M\tilde{R}_{ca} & \overleftrightarrow{I}_a & \beta^{B^t}_a \end{bmatrix} \cdot \begin{bmatrix} \vec{v}_a \\ \\ \vec{\omega}^B \\ \\ \dot{\xi}^B_a \end{bmatrix} \tag{3-28}$$

$$T = \frac{1}{2}\begin{bmatrix} \vec{v}_a & \vec{\omega}^B & \dot{\xi}^{B^t}_a \end{bmatrix} \cdot \begin{bmatrix} M\overleftrightarrow{E} & M\tilde{R}^t_{ca} & \alpha^{B^t}_a \\ M\tilde{R}_{ca} & \overleftrightarrow{I}_a & \beta^{B^t}_a \\ \alpha^B_a & \beta^B_a & \gamma^B_a \end{bmatrix} \cdot \begin{bmatrix} \vec{v}_a \\ \vec{\omega}^B \\ \dot{\xi}^B_a \end{bmatrix} \tag{3-29}$$

$$T = \begin{bmatrix} \vec{F} & \vec{L}_a & k^{B^t}_a \end{bmatrix} \cdot \begin{bmatrix} \vec{v}_a \\ \\ \vec{\omega}^B \\ \\ \dot{\xi}^B_a \end{bmatrix} \tag{3-30}$$

In addition, we have the relationships

$$\frac{\partial T}{\partial \vec{v}_a} = \vec{P} \tag{3-31}$$

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* See Appendix B

$$\frac{\partial T}{\partial \vec{\omega}^B} = \vec{H}_a \tag{3-32}$$

where here the partial derivatives of T mean the partial derivatives of the function of $\vec{v}_a$, $\vec{\omega}^B$, and $\dot{\xi}_a^B$ (and of the internal generalized coordinates ξ_{a1}^B, ξ_{a2}^B, $\cdots$, ξ_{an}^B) given in Equation (29).

We have now achieved our objective of expressing the fundamental quantities in terms of the velocities of the 6 external degrees of freedom plus the velocities of the n internal degrees of freedom. However, so far, we only have the external momenta $\vec{P}$ and $\vec{H}_a$; we would also like to have n internal generalized momenta. Since the external momenta can be obtained by partial differentiation of the kinetic energy with respect to the angular velocities, we are led to define the n internal generalized momenta by partial differentiation of T with respect to $\dot{\xi}_a^B$. Denoting the internal generalized momentum by g_a^B we thus have

$$\frac{\partial T}{\partial \dot{\xi}_a^B} = g_a^B = \alpha_a^B \cdot \vec{v}_a + \beta_a^B \cdot \vec{\omega}^B + \gamma_a^B \dot{\xi}_a^B \tag{3-33}$$

Combining Equations (28) and (33) now yields

$$\begin{bmatrix} \vec{P} \\ \vec{H}_a \\ g_a^B \end{bmatrix} = \begin{bmatrix} M\overleftrightarrow{E} & M\widetilde{R}_{ca}^t & {\alpha_a^B}^t \\ M\widetilde{R}_{ca} & \overleftrightarrow{I}_a & {\beta_a^B}^t \\ \alpha_a^B & \beta_a^B & \gamma_a^B \end{bmatrix} \cdot \begin{bmatrix} \vec{v}_a \\ \vec{\omega}^B \\ \dot{\xi}_a^B \end{bmatrix} \tag{3-34}$$

$$T = \frac{1}{2}\begin{bmatrix} \vec{v}_a & \vec{\omega}^B & \dot{\xi}^B_a \end{bmatrix}^t \begin{bmatrix} \vec{P} \\ \vec{H}_a \\ g^B_a \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \vec{P} & \vec{H}_a & g^B_a \end{bmatrix}^t \cdot \begin{bmatrix} \vec{v}_a \\ \vec{\omega}^B \\ \dot{\xi}^B_a \end{bmatrix} \qquad (3\text{-}35)$$

We now have a full set of external and internal velocities, plus a full set of external and internal momenta. A full set of dynamics equations is a set of equations for $\dot{\vec{P}}$, $\dot{\vec{H}}_a$, and $\dot{g}^B_a$, or a set of equations for $\dot{\vec{v}}_a$, $\dot{\vec{\omega}}^B$, and $\ddot{\xi}^B_a$. Equations (2-71) and (2-72) give us expressions for $\dot{\vec{P}}$ and $\dot{\vec{H}}_a$, respectively; this will be a full set of dynamics equations as soon as we get an equation for $\dot{g}^B_a$ (see Section V, page 65).

Recall that we defined $\vec{P}$ and $\vec{H}_a$ in terms of $\vec{p}_i$ (Equations (2-6) and (2-40), respectively). Thus, we were able to obtain equations for $\dot{\vec{P}}$ and $\dot{\vec{H}}_a$ by simply differentiating these defining equations with respect to time, and then using Newton's law to replace $\dot{\vec{p}}_i$ with $\vec{f}_i$. In the next section we will define g^B_a as being given by $\sum_i \Phi^B_{ia} \cdot \vec{p}_i$. From this definition we will obtain $\dot{g}^B_a$ as $\sum_i \dot{\Phi}^B_{ia} \cdot \vec{p}_i + \sum_i \Phi^B_{ia} \cdot \dot{\vec{p}}_i$. When we replace $\dot{\vec{p}}_i$ with $\vec{f}_i$ we notice that the second of these summations is simply k^B_a as defined in Equation (22). We will then have our equation for $\dot{g}^B_a$ as soon as we put $\sum_i \dot{\Phi}^B_{ia} \cdot \vec{p}_i$ in a more convenient form which does not involve an explicit summation over particles. Of course, we also have to show that the definition of g^B_a as $\sum_i \Phi^B_{ia} \cdot \vec{p}_i$ is equivalent to the definition of g^B_a as in Equation (33).

IV. TRANSFORMATION OPERATOR EQUATIONS*

We will now use the transformation operator approach to derive the desired full set of equations of motion for our deformable body. The transformation operator formalism has been documented in References (66) and (68) and therefore we will give here only a brief summary in a form most suitable for application to a deformable body. It is interesting to note that the entire formalism, which is based on Kron's method of subspaces[70-73], is essentially a generalization of linear matrix structural analysis.[83-84] The generalization consists mainly of starting out with coordinate dependent velocity transformations rather than with linear and constant coordinate transformations. In general, coordinate transformations are nonlinear, but the corresponding velocity transformations are always linear.

A. BASIC FORMALISM

The equations of motion for a dynamical system with a finite number of degrees of freedom can be put in the form

$$\dot{G} + X = K \tag{4-1}$$

where G is the system momentum, K is the system force, and X is an extra term which is quadratic in the momentum. In general X also depends on the coordinates. The momentum G is linearly related to the system velocity σ:

$$G = \mu \cdot \sigma \tag{4-2}$$

where μ is the system mass. Taking the time derivative of

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* The final equations obtained in this section are summarized in Section V, page 65. The reader may wish to look at Section V before reading Section IV.

Equation (2) and then substituting into Equation (1) yields

$$\mu \cdot \dot{\sigma} + Y = K \tag{4-3}$$

where

$$Y = \dot{\mu} \cdot \sigma + X \tag{4-4}$$

Equation (1) is a "momentum formulation" whereas Equation (3) is a "velocity formulation." The principal difference between the momentum and velocity formulations is that in the momentum formulation we have the extra term X whereas in the velocity formulation we have the extra term Y. Often it is simpler to obtain X than it is to obtain Y (this is certainly the case if to obtain Y, we first obtain X as indicated in Equation (4)).[74-76]

In addition to Equations (1) to (4), we also have the kinetic energy expressions

$$T = \frac{1}{2}\sigma^t \cdot \mu \cdot \sigma = \frac{1}{2}G^t \cdot \sigma \tag{4-5}$$

$$\dot{T} = K^t \cdot \sigma \tag{4-6}$$

$$\frac{\partial T}{\partial \sigma} = G \tag{4-7}$$

When performing the partial differentiation of T with respect to σ we consider T to be the quadratic function of σ given in Equation (5).

The system mass is positive definite and symmetric, and therefore it has a positive definite symmetric inverse ν :

$$\mu^{-1} = \nu \tag{4-8}$$

Therefore, Equation (2) can be inverted as follows

$$\sigma = \nu \cdot G \tag{4-9}$$

Consequently, the kinetic energy can also be expressed as

$$T = \frac{1}{2} G^t \cdot \sigma = \frac{1}{2} G^t \cdot \nu \cdot G \tag{4-10}$$

From this follows the relationship

$$\frac{\partial T}{\partial G} = \sigma \tag{4-11}$$

where here we consider T to be the quadratic function of G given in Equation (10).

We now make a linear velocity transformation

$$\sigma = A \cdot \overline{\sigma} \tag{4-12}$$

where $\overline{\sigma}$ is a new system velocity and A is the transformation operator. In conjunction with this velocity transformation, we also make the following momentum and force transformations

$$\overline{G} = A^t \cdot G \tag{4-13}$$

$$\overline{K} = A^t \cdot K \tag{4-14}$$

We also make a congruence transformation on the mass:

$$\overline{\mu} = A^t \cdot \mu \cdot A \tag{4-15}$$

Substituting Equation (12) into Equations (5) and (6) and then making use of Equations (13) to (15) yields

$$T = \frac{1}{2}\overline{\sigma}^t \cdot \overline{\mu} \cdot \overline{\sigma} = \frac{1}{2}\overline{G}^t \cdot \overline{\sigma} \tag{4-16}$$

$$\dot{T} = \overline{K}^t \cdot \overline{\sigma} \tag{4-17}$$

Thus $\overline{\sigma}$, $\overline{\mu}$, $\overline{G}$, and $\overline{K}$ are properly defined in order to keep the functional form of T and $\dot{T}$ invariant.

Differentiating $\overline{G}$ with respect to time yields

$$\dot{\overline{G}} + \overline{X} = \overline{K} \tag{4-18}$$

where

$$\overline{X} = A^t \cdot X - \dot{A}^t \cdot G \tag{4-19}$$

We also find that $\overline{G}$ and $\overline{\sigma}$ are related as follows

$$\overline{G} = \overline{\mu} \cdot \overline{\sigma} = \frac{\partial T}{\partial \overline{\sigma}} \tag{4-20}$$

Substituting Equation (20) into Equation (18) yields

$$\overline{\mu} \cdot \dot{\overline{\sigma}} + \overline{Y} = \overline{K} \tag{4-21}$$

where

$$\overline{Y} = \dot{\overline{\mu}} \cdot \overline{\sigma} + \overline{X} \tag{4-22}$$

We get an alternate expression for $\overline{Y}$ by substituting Equation (12) into (3) and then multiplying the resulting Equation (3) from the left by A^t. The resulting $\overline{Y}$ is

$$\overline{Y} = A^t \cdot Y + A^t \cdot \mu \cdot \dot{A} \cdot \overline{\sigma} \tag{4-23}$$

If Equation (22) is used to obtain $\overline{Y}$, then it obviously requires more effort to obtain $\overline{Y}$ than to obtain only $\overline{X}$. However, if Equation (23) is used for $\overline{Y}$, it is not clear whether it is simpler to find this $\overline{Y}$ or the $\overline{X}$ given by Equation (19). In fact, by writing $G = \mu \cdot \sigma = \mu \cdot A \cdot \overline{\sigma}$ we can put $\overline{X}$ in the form

$$\overline{X} = A^t \cdot X - \dot{A}^t \cdot \mu \cdot A \cdot \overline{\sigma} \tag{4-24}$$

Comparing Equations (23) and (24), we note that both $\overline{X}$ and $\overline{Y}$ appear to be equally complicated.

In general, the transformation operator A is not invertible, but it is always one-to-one (alternatively, it always has full column rank). Therefore, A has a left inverse[85-86] B so that

$$B \cdot A = \overline{I} \tag{4-25}$$

where $\overline{I}$ is the identity of the same dimension as $\overline{\sigma}$ (and also of the same dimension as $\overline{G}$, $\overline{K}$, $\overline{X}$, and $\overline{Y}$). From Equation (25) it follows that

$$A^t \cdot B^t = \overline{I} \tag{4-26}$$

Thus, A^t has a right inverse B^t. However, in general, A does not have a right inverse, and therefore A^t in general does not have a left inverse. Consequently, Equation (12) can be solved for $\overline{\sigma}$:

$$\overline{\sigma} = B \cdot \sigma \tag{4-27}$$

but Equations (13) and (14) cannot similiarly be solved for G and K in terms of $\overline{G}$ and $\overline{K}$, respectively. Similiarly, Equation (15) cannot be solved for μ in terms of $\overline{\mu}$.

Since μ is positive definite symmetric and since A is one-to-one, it follows from Equation (15) that $\overline{\mu}$ is also positive definite symmetric. Therefore $\overline{\mu}$ has a *positive definite* symmetric inverse $\overline{\nu}$:

$$\overline{\mu}^{-1} = \overline{\nu} \tag{4-28}$$

Thus $\overline{\sigma}$ can be expressed as follows:

$$\overline{\sigma} = \overline{\nu} \cdot \overline{G} = \frac{\partial T}{\partial \overline{G}} \tag{4-29}$$

It can be shown from general tensorial considerations that there exists a $\overline{C}$ such that*

$$\dot{\overline{\mu}} = \overline{C} \cdot \overline{\mu} + \overline{\mu} \cdot \overline{C}^t \tag{4-30}$$

$$\overline{X} = -\overline{C} \cdot \overline{G} = -\overline{C} \cdot \overline{\mu} \cdot \overline{\sigma} \tag{4-31}$$

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* $\overline{C}$ is linear in $\overline{\sigma}$ and in the Christoffel symbols (which generally are functions of the coordinates).

$$\overline{Y} = \overline{\mu} \cdot \overline{C}^t \cdot \overline{\sigma} \tag{4-32}$$

Now define $\overline{D}$ by

$$\overline{D} = \overline{C} \cdot \overline{\mu} \tag{4-33}$$

Then Equations (30) to (32) become

$$\dot{\overline{\mu}} = \overline{D} + \overline{D}^t \tag{4-34}$$

$$\overline{X} = -\overline{D} \cdot \overline{\sigma} \tag{4-35}$$

$$\overline{Y} = \overline{D}^t \cdot \overline{\sigma} \tag{4-36}$$

where we have made use of the fact that $\overline{\mu}$ is symmetric ($\overline{\mu} = \overline{\mu}^t$). From these equations it is particularly evident that

$$\overline{Y} - \overline{X} = \dot{\overline{\mu}} \cdot \overline{\sigma} \tag{4-37}$$

From Equations (35) and (36) it is evident that if we obtain $\overline{X}$ and $\overline{Y}$ by first obtaining $\overline{D}$, then $\overline{X}$ and $\overline{Y}$ are obtained with equal difficulty.

B. APPLICATION TO A DEFORMABLE BODY

In order to use the transformation operator formalism, we must start out with a set of σ, μ, G, K, X, and Y which satisfy Equations (1) to (3). Then we must introduce a new velocity $\overline{\sigma}$ and a transformation operator A. The transformation operator formalism then does the rest.

We consider a system of N particles and we let σ be a column matrix of N vectors, whose i^{th} element (vector) is $\vec{v}_i$, the velocity of the i^{th} particle:

$$\sigma = \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \cdot \\ \cdot \\ \cdot \\ \vec{v}_N \end{bmatrix} \tag{4-38}$$

We define μ to be an $N \times N$ matrix of dyadics which is diagonal and whose i^{th} diagonal element is $m_i\overleftrightarrow{E}$, where m_i is the mass of the i^{th} particle:

$$\mu = \begin{bmatrix} m_1\overleftrightarrow{E} & & & \\ & m_2\overleftrightarrow{E} & & \\ & & \ddots & \\ & & & m_N\overleftrightarrow{E} \end{bmatrix} \tag{4-39}$$

where the off-diagonal elements of μ are the zero dyadic. Since $G = \mu \cdot \sigma$ we see that G must be a column matrix of N vectors, whose i^{th} element is $\vec{p}_i$, the linear momentum of the i^{th} particle:

$$G = \begin{bmatrix} \vec{p}_1 \\ \vec{p}_2 \\ \cdot \\ \cdot \\ \cdot \\ \vec{p}_N \end{bmatrix} \tag{4-40}$$

Next, we define K to be a column matrix of N vectors, whose i^{th} element is $\vec{f}_i$, the force on the i^{th} particle:

$$K = \begin{bmatrix} \vec{f}_1 \\ \vec{f}_2 \\ \cdot \\ \cdot \\ \cdot \\ \vec{f}_N \end{bmatrix} \tag{4-41}$$

Since we have $\dot{\vec{p}}_i = \vec{f}_i$ we obviously have X equal to zero:

$$X = 0 \tag{4-42}$$

Thus, X is a column matrix of N zero vectors. Similiarly, since $m_i \dot{\vec{v}}_i = \vec{f}_i$, we also have Y equal to zero:

$$Y = 0 \tag{4-43}$$

and therefore Y is also a column matrix of N zero vectors.

Thus, the equations of motion for the system of particles are

$$\dot{G} = K \qquad \text{(since } X = 0\text{)} \tag{4-44}$$

$$G = \mu \cdot \sigma \tag{4-45}$$

These equations represent the momentum formulation. The velocity formulation equation is

$$\mu \cdot \dot{\sigma} = K \qquad \text{(since } Y = 0\text{)} \tag{4-46}$$

We now define $\overline{\sigma}$ as follows:

$$\overline{\sigma} = \begin{bmatrix} \vec{v}_a \\ \vec{\omega}^B \\ \dot{\xi}^B_a \end{bmatrix} \tag{4-47}$$

where $\vec{v}_a$ is the inertial linear velocity of an arbitrary point a as introduced in Section II , $\vec{\omega}^B$ is the inertial angular velocity of an arbitrary frame B as introduced in Section II , and $\dot{\xi}^B_a$ is an n element column matrix whose j^{th} element is $\dot{\xi}^B_{aj}$, which is the time derivative of the j^{th} internal generalized coordinate.

We determine the transformation operator A by combining the following three equations (see Equations (2-26), (2-61), (3-1), and (3-9))

$$\vec{v}_i = \vec{v}_a + \dot{\vec{R}}_{ia} \tag{4-48}$$

$$\dot{\vec{R}}_{ia} = \vec{u}^{B}_{ia} + \tilde{R}^{t}_{ia} \cdot \vec{\omega}^{B} \qquad (4\text{-}49)$$

$$\vec{u}^{B}_{ia} = \sum_{j=1}^{n} \vec{\Phi}^{B}_{iaj}\, \dot{\xi}^{B}_{aj} = \Phi^{B^t}_{ia}\, \dot{\xi}^{B}_{a} \qquad (4\text{-}50)$$

Thus we have

$$\vec{v}_i = \vec{v}_a + \tilde{R}^{t}_{ia} \cdot \vec{\omega}^{B} + \Phi^{B^t}_{ia}\, \dot{\xi}^{B}_{a} \qquad (4\text{-}51)$$

$$= \begin{bmatrix} \overleftrightarrow{E} & \tilde{R}^{t}_{ia} & \Phi^{B^t}_{ia} \end{bmatrix} \cdot \begin{bmatrix} \vec{v}_a \\ \vec{\omega}^{B} \\ \dot{\xi}^{B}_{a} \end{bmatrix}$$

Thus, A is given by

$$A = \begin{bmatrix} \overleftrightarrow{E} & \tilde{R}^{t}_{1a} & \Phi^{B^t}_{1a} \\ \overleftrightarrow{E} & \tilde{R}^{t}_{2a} & \Phi^{B^t}_{2a} \\ \vdots & \vdots & \vdots \\ \overleftrightarrow{E} & \tilde{R}^{t}_{Na} & \Phi^{B^t}_{Na} \end{bmatrix} \qquad (4\text{-}52)$$

$\bar{G}$ is now defined by

$$\overline{G} = A^t \cdot G = \begin{bmatrix} \overleftrightarrow{E} & \overleftrightarrow{E} & \cdots & \overleftrightarrow{E} \\ \tilde{R}_{1a} & \tilde{R}_{2a} & \cdots & \tilde{R}_{Na} \\ \Phi^B_{1a} & \Phi^B_{2a} & \cdots & \Phi^B_{Na} \end{bmatrix} \begin{bmatrix} \vec{p}_1 \\ \vec{p}_2 \\ \cdot \\ \cdot \\ \cdot \\ \vec{p}_N \end{bmatrix} \tag{4-53}$$

$$= \begin{bmatrix} \sum_i \vec{p}_i \\ \sum_i \tilde{R}_{ia} \cdot \vec{p}_i \\ \sum_i \Phi^B_{ia} \cdot \vec{p}_i \end{bmatrix}$$

From Equation (2-6) we see that the first element of $\overline{G}$ is $\vec{P}$, the total system linear momentum. From Equation (2-40) we see the second element of $\overline{G}$ is $\vec{H}_a$, the total system angular momentum about a. We will denote the third element of $\overline{G}$ by g^B_a. Later, we will see that this is the same quantity which we introduced in Equation (3-33) and which we called the internal generalized momentum. Thus, $\overline{G}$ is given by

$$\overline{G} = \begin{bmatrix} \vec{P} \\ \vec{H}_a \\ g^B_a \end{bmatrix} \tag{4-54}$$

$\overline{K}$ is obtained similarly:

$$\overline{K} = A^t \cdot K = \begin{bmatrix} \sum_i \vec{f}_i \\ \sum_i \widetilde{R}_{ia} \cdot \vec{f}_i \\ \sum_i \Phi^B_{ia} \cdot \vec{f}_i \end{bmatrix} = \begin{bmatrix} \vec{F} \\ \vec{L}_a \\ k^B_a \end{bmatrix} \tag{4-55}$$

where we have made use of Equations(2-17),(2-46), and(3-22). Thus, the first element of $\overline{K}$ is the total (external) force on the system, the second element of $\overline{K}$ is the total (external) torque on the system about a, and k^B_a is a quantity which we can call the force on the internal degrees of freedom. If we split $\vec{f}_i$ up into an external part $\vec{f}^E_i$ (whose origin is from outside of the deformable body) and an internal part $\vec{f}^I_i$ (due to coupling among the mass points of the body), then we can write

$$k^B_a = k^{BE}_a + k^{BI}_a \tag{4-56}$$

where

$$k^{BE}_a = \sum_i \Phi^B_{ia} \cdot \vec{f}^E_i \tag{4-57}$$

$$k^{BI}_a = \sum_i \Phi^B_{ia} \cdot \vec{f}^I_i \tag{4-58}$$

$\overline{\mu}$ is obtained as follows

$$\overline{\mu} = A^t \cdot \mu \cdot A = \begin{bmatrix} \sum_i m_i \overleftrightarrow{E} & \sum_i m_i \widetilde{R}^t_{ia} & \sum_i m_i \Phi^{B^t}_{ia} \\ \sum_i m_i \widetilde{R}_{ia} & \sum_i m_i \widetilde{R}_{ia} \cdot \widetilde{R}^t_{ia} & \sum_i m_i \widetilde{R}_{ia} \cdot \Phi^{B^t}_{ia} \\ \sum_i m_i \Phi^B_{ia} & \sum_i m_i \Phi^B_{ia} \cdot \widetilde{R}^t_{ia} & \sum_i m_i \Phi^B_{ia} \cdot \Phi^{B^t}_{ia} \end{bmatrix}$$

$$= \begin{bmatrix} M\overleftrightarrow{E} & M\widetilde{R}^t_{ca} & \alpha^{B^t}_a \\ M\widetilde{R}_{ca} & \overleftrightarrow{I}_a & \beta^{B^t}_a \\ \alpha^B_a & \beta^B_a & \gamma^B_a \end{bmatrix} \tag{4-59}$$

where we have introduced M from Equation(2-3), $M\vec{R}_{ca}$ from Equation(2-25), $\overleftrightarrow{I}_a$ from Equation(2-66), α^B_a from Equation(3-17), β^B_a from Equation(3-19), and γ^B_a from Equation(3-21).

The equation $\overline{G} = \overline{\mu} \cdot \overline{\sigma}$ is now precisely Equation(3-34) given in Section III. This proves the statement that g^B_a as defined in this section is the same quantity as defined in Section III:

$$g^B_a = \sum_i \Phi^B_{ia} \cdot \vec{p}_i = \alpha^B_a \cdot \vec{v}_a + \beta^B_a \cdot \vec{\omega}^B + \gamma^B_a \dot{\xi}^B_a \tag{4-60}$$

If the arbitrary point a coincides with the total center of mass c, then $\vec{P} = M\vec{v}_c$ and $\vec{R}_{cc} = 0$ imply that

$$\alpha^B_c = \sum_i m_i \Phi^B_{ic} = 0 \tag{4-61}$$

Hence, in this case, the internal generalized momentum is given by

$$g_c^B = \sum_i \Phi_{ic}^B \cdot \vec{p}_i = \beta_c^B \cdot \vec{\omega}^B + \gamma_c^B \, \dot{\xi}_c^B \tag{4-62}$$

where

$$\beta_c^B = \sum_i m_i \Phi_{ic}^B \cdot \tilde{R}_{ic}^t \tag{4-63}$$

$$\gamma_c^B = \sum_i m_i \Phi_{ic}^B \cdot \Phi_{ic}^{B^t} \tag{4-64}$$

Note that after the arbitrary point a has been specified (say, by letting a coincide with c), we still have arbitrariness in Φ_{ia}^B, g_a^B and $\dot{\xi}_a^B$ due to the arbitrariness of the frame B.

C. THE FORM OF $\overline{X}$ AND $\overline{Y}$ FOR A DEFORMABLE BODY

So far, our application of the transformation operator formalism to a deformable body has not really yielded much more than could be obtained from linear matrix structural analysis. However, it should be noted that our formalism has additional flexibility because we are actually using a coordinate dependent transformation matrix A [Equation (52)], and this transformation matrix has some elements which are dyadics and some elements which are row matrices of vectors. The key ingredients in a set of exact dynamics equations is the extra term $\overline{X}$ or $\overline{Y}$, which depends quadratically on the momentum or velocity. The $\overline{X}$ and $\overline{Y}$ terms cannot be obtained from linear matrix structural analysis because these terms depend on the time derivative of

the transformation operator A. We now turn to determining the form of $\overline{X}$ and $\overline{Y}$ for the deformable body.

Since in the present application both X and Y are zero, we have from Equations (23) and (24)

$$\overline{X} = -\dot{A}^t \cdot G = -\dot{A}^t \cdot \mu \cdot A \cdot \overline{\sigma} \qquad \text{(since } X = 0\text{)} \tag{4-65}$$

$$\overline{Y} = A^t \cdot \mu \cdot \dot{A} \cdot \overline{\sigma} \qquad \text{(since } Y = 0\text{)} \tag{4-66}$$

Comparing Equations (65) and (66) with Equations (34) to (36), we see that in this case

$$\overline{D} = \dot{A}^t \cdot \mu \cdot A \tag{4-67}$$

$$= \begin{bmatrix} \overleftrightarrow{O} & \overleftrightarrow{O} & 0 \\ \sum_i m_i \dot{\tilde{R}}_{ia} & \sum_i m_i \dot{\tilde{R}}_{ia} \cdot \tilde{R}^t_{ia} & \sum_i m_i \dot{\tilde{R}}_{ia} \cdot \Phi^{B^t}_{ia} \\ \sum_i m_i \dot{\Phi}^B_{ia} & \sum_i m_i \dot{\Phi}^B_{ia} \cdot \tilde{R}^t_{ia} & \sum_i m_i \dot{\Phi}^B_{ia} \cdot \Phi^{B^t}_{ia} \end{bmatrix}$$

where $\overleftrightarrow{O}$ is the zero dyadic, and 0 represents a row matrix of n zero vectors. We can write $\overline{D}$ more compactly as follows:

$$\overline{D} = \begin{bmatrix} \overleftrightarrow{O} & \overleftrightarrow{O} & 0 \\ M\dot{\tilde{R}}_{ca} & \overline{D}_{22} & \overline{D}_{23} \\ \dot{\alpha}^B_a & \overline{D}_{32} & \overline{D}_{33} \end{bmatrix} \tag{4-68}$$

where

$$\bar{D}_{22} = \sum_i m_i \dot{\tilde{R}}_{ia} \cdot \tilde{R}^t_{ia} \tag{4-69}$$

$$\bar{D}_{32} = \sum_i m_i \dot{\Phi}^B_{ia} \cdot \tilde{R}^t_{ia} \tag{4-70}$$

$$\bar{D}_{23} = \sum_i m_i \dot{\tilde{R}}_{ia} \cdot \Phi^{B^t}_{ia} \tag{4-71}$$

$$\bar{D}_{33} = \sum_i m_i \dot{\Phi}^B_{ia} \cdot \Phi^{B^t}_{ia} \tag{4-72}$$

Hence

$$\bar{D} + \bar{D}^t = \begin{bmatrix} \overleftrightarrow{O} & M\dot{\tilde{R}}^t_{ca} & \dot{\alpha}^{B^t}_a \\ M\dot{\tilde{R}}_{ca} & \bar{D}_{22} + \bar{D}_{22}{}^t & \bar{D}_{23} + \bar{D}^t_{32} \\ \dot{\alpha}^B_a & \bar{D}_{32} + \bar{D}_{23}{}^t & \bar{D}_{33} + \bar{D}_{33}{}^t \end{bmatrix} \tag{4-73}$$

Taking the time derivative of $\bar{\mu}$ yields

$$\dot{\bar{\mu}} = \begin{bmatrix} \overleftrightarrow{O} & M\dot{\tilde{R}}^t_{ca} & \dot{\alpha}^{B^t}_a \\ M\dot{\tilde{R}}_{ca} & \dot{\overleftrightarrow{I}}_a & \dot{\beta}^{B^t}_a \\ \dot{\alpha}^B_a & \dot{\beta}^B_a & \dot{\gamma}^B_a \end{bmatrix} \tag{4-74}$$

Since $\dot{\bar{\mu}} = \bar{D} + \bar{D}^t$ we see that we now have

$$\dot{\overleftrightarrow{I}}_a = \bar{D}_{22} + \bar{D}_{22}{}^t = \dot{\overleftrightarrow{I}}^t_a \tag{4-75}$$

$$\dot{\beta}_a^B = \bar{D}_{32} + \bar{D}_{23}{}^t \tag{4-76}$$

$$\dot{\gamma}_a^B = \bar{D}_{33} + \bar{D}_{33}{}^t \tag{4-77}$$

We can now write $\bar{X}$ and $\bar{Y}$ as

$$\bar{X} = -\bar{D} \cdot \bar{\sigma} = - \begin{bmatrix} \vec{0} \\ \\ M\dot{\tilde{R}}_{ca} \cdot \vec{v}_a + \bar{D}_{22} \cdot \vec{\omega}^B + \bar{D}_{23}\, \dot{\xi}_a^B \\ \\ \dot{\alpha}_a^B \cdot \vec{v}_a + \bar{D}_{32} \cdot \vec{\omega}^B + \bar{D}_{33}\, \dot{\xi}_a^B \end{bmatrix} \tag{4-78}$$

$$\bar{Y} = \bar{D}^t \cdot \bar{\sigma} = \begin{bmatrix} M\dot{\tilde{R}}^t_{ca} \cdot \vec{\omega}^B + \dot{\alpha}_a^{B^t}\, \dot{\xi}_a^B \\ \\ \bar{D}_{22}{}^t \cdot \vec{\omega}^B + \bar{D}_{32}{}^t\, \dot{\xi}_a^B \\ \\ \bar{D}_{23}{}^t \cdot \vec{\omega}^B + \bar{D}_{33}{}^t\, \dot{\xi}_a^B \end{bmatrix} \tag{4-79}$$

We will now show that the second element of $\bar{X}$ is $\vec{v}_a \times \vec{P}$ as required by Equation(2-49). Note that

$$\begin{aligned} -M\dot{\tilde{R}}_{ca} \cdot \vec{v}_a &= -M(\vec{v}_c - \vec{v}_a) \times \vec{v}_a = -M\vec{v}_c \times \vec{v}_a \\ &= \vec{v}_a \times \vec{P} \end{aligned} \tag{4-80}$$

Next we show that

$$\bar{D}_{22} \cdot \vec{\omega}^B + \bar{D}_{23}\, \dot{\xi}_a^B = \vec{0} \tag{4-81}$$

From Equations (69) and (71), we see that this requires

$$\sum_i m_i \dot{\tilde{R}}_{ia} \cdot \tilde{R}^t_{ia} \cdot \vec{\omega}^B + \sum_i m_i \dot{\tilde{R}}_{ia} \cdot \Phi^{B^t}_{ia} \dot{\xi}^B_a = \vec{0} \tag{4-82}$$

Factoring out the common term $m_i \dot{\tilde{R}}_{ia}$ and introducing $\dot{\vec{R}}_{ia}$ from Equations (48) and (51) allows us to write Equation (82) as

$$\sum_i m_i \dot{\tilde{R}}_{ia} \cdot \dot{\vec{R}}_{ia} = \vec{0} \tag{4-83}$$

This equation is obviously satisfied and therefore Equation (81) is established. Thus, Equation (78) can be written as

$$\overline{X} = \begin{bmatrix} \overline{X}_{\vec{v}_a} \\ \overline{X}_{\vec{\omega}^B} \\ \overline{X}_{\dot{\xi}^B_a} \end{bmatrix} = - \begin{bmatrix} \vec{0} \\ M\dot{\tilde{R}}_{ca} \cdot \vec{v}_a \\ \dot{\alpha}^B_a \cdot \vec{v}_a + \overline{D}_{32} \cdot \vec{\omega}^B + \overline{D}_{33} \dot{\xi}^B_a \end{bmatrix} = \begin{bmatrix} \vec{0} \\ \vec{v}_a \times \vec{P} \\ \overline{X}_{\dot{\xi}^B_a} \end{bmatrix} \tag{4-84}$$

where we have introduced the symbols $\overline{X}_{\vec{v}_a}$, $\overline{X}_{\vec{\omega}^B}$, and $\overline{X}_{\dot{\xi}^B_a}$ for the elements of $\overline{X}$. Note that $\overline{X}_{\dot{\xi}^B_a}$ is an n element column matrix of scalars; the j^{th} element of this matrix is denoted by $\overline{X}_{\dot{\xi}^B_{aj}}$.

The expression for $\overline{Y}$ given in Equation (79) cannot be simplified. However, we can get an alternate expression for $\overline{Y}$ from $\overline{Y} = \dot{\overline{\mu}} \cdot \overline{\sigma} + \overline{X}$. Using Equation (74) for $\dot{\overline{\mu}}$ and Equation (84)

for $\overline{X}$ yields

$$\overline{Y} = \begin{bmatrix} M\dot{\tilde{\vec{R}}}{}^{t}_{ca} \cdot \vec{\omega}^B + \dot{\alpha}^{B^t}_{a} \, \xi^B_a \\ \\ M\dot{\tilde{\vec{R}}}_{ca} \cdot \vec{v}_a + \overleftrightarrow{I}_a \cdot \vec{\omega}^B + \dot{\beta}^{B^t}_{a} \, \dot{\xi}^B_a \\ \\ \dot{\alpha}^B_a \cdot \vec{v}_a + \dot{\beta}^B_a \cdot \vec{\omega}^B + \dot{\gamma}^B_a \, \dot{\xi}^B_a \end{bmatrix} + \begin{bmatrix} \vec{0} \\ \\ -M\dot{\tilde{\vec{R}}}_{ca} \cdot \vec{v}_a \\ \\ \overline{X}_{\dot{\xi}^B_a} \end{bmatrix} \tag{4-85}$$

or

$$\overline{Y} = \begin{bmatrix} \overline{Y}_{\vec{v}_a} \\ \\ \overline{Y}_{\vec{\omega}^B} \\ \\ \overline{Y}_{\dot{\xi}^B_a} \end{bmatrix} = \begin{bmatrix} M\dot{\tilde{\vec{R}}}{}^{t}_{ca} \cdot \vec{\omega}^B + \dot{\alpha}^{B^t}_{a} \, \dot{\xi}^B_a \\ \\ \overleftrightarrow{I}_a \cdot \vec{\omega}^B + \dot{\beta}^{B^t}_{a} \, \dot{\xi}^B_a \\ \\ \overline{D}_{23}{}^{t} \cdot \vec{\omega}^B + \overline{D}_{33}{}^{t} \, \dot{\xi}^B_a \end{bmatrix} \tag{4-86}$$

where we have introduced the symbols $\overline{Y}_{\vec{v}_a}$, $\overline{Y}_{\vec{\omega}^B}$, and $\overline{Y}_{\dot{\xi}^B_a}$ for the elements of $\overline{Y}$. $\overline{Y}_{\dot{\xi}^B_a}$ is an n element column matrix of scalars; the j^{th} element of this matrix is denoted by $\overline{Y}_{\dot{\xi}^B_{aj}}$.

Lagrange's equation for the internal generalized coordinates takes the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\xi}^B_a}\right) - \frac{\partial T}{\partial \xi^B_a} = k^B_a \tag{4-87}$$

The first term on the left is $\dot{g}_a^B$. Therefore, the second term on the left is $\overline{X}_{\dot{\xi}_a^B}$. Thus, an alternate expression for $\overline{X}_{\dot{\xi}_a^B}$ is

$$\overline{X}_{\dot{\xi}_a^B} = - \frac{\partial T}{\partial \xi_a^B} \tag{4-88}$$

We will return to this equation after we get explicit expressions for $\overline{D}$ and $\overline{X}$.

D. EXPLICIT EXPRESSIONS FOR $\overline{D}$, $\overline{X}$, AND $\overline{Y}$ FOR A DEFORMABLE BODY

The expressions for $\overline{D}$, $\overline{X}$, and $\overline{Y}$ given above, show the general form of these quantities, but they do not show explicitly the dependence of these terms on the velocities $\vec{v}_a$, $\vec{\omega}^B$, and $\dot{\xi}_a^B$. From Equation (67), we note that in order to get explicit expressions for $\overline{D}$, we need to express $\dot{\tilde{R}}_{ia}$ and $\dot{\Phi}_{ia}^B$ in terms of $\vec{v}_a$, $\vec{\omega}^B$, and $\dot{\xi}_a^B$.

From Equations (2-58), (2-59), (3-1), and (3-9) we have

$$\dot{\vec{R}}_{ia} = \dot{\xi}_a^{B^t} \Phi_{ia}^B + \tilde{\omega}^B \cdot \vec{R}_{ia} \tag{4-89}$$

and from this follows

$$\dot{\tilde{R}}_{ia} = \dot{\xi}_a^{B^t} \tilde{\Phi}_{ia}^B + \tilde{\omega}^B \cdot \tilde{R}_{ia} - \tilde{R}_{ia} \cdot \tilde{\omega}^B \tag{4-90}$$

where $\tilde{\Phi}_{ia}^B$ is formed from Φ_{ia}^B by replacing all vector elements by the skew-symmetric dyadic of these vectors. Thus,

$$\tilde{\Phi}^B_{ia} = \begin{bmatrix} \tilde{\Phi}^B_{ia1} \\ \tilde{\Phi}^B_{ia2} \\ \cdot \\ \cdot \\ \cdot \\ \tilde{\Phi}^B_{ian} \end{bmatrix} \tag{4-91}$$

Taking the transpose of Equation (90) yields

$$\dot{\tilde{R}}^t_{ia} = \tilde{\Phi}^{B^t}_{ia} \dot{\boldsymbol{\xi}}^B_a - \tilde{R}^t_{ia} \cdot \tilde{\omega}^B + \tilde{\omega}^B \cdot \tilde{R}^t_{ia} \tag{4-92}$$

where we have changed signs in the second and third terms by using $\tilde{\omega}^{B^t} = -\tilde{\omega}^B$. It should be noted that $\tilde{\Phi}^{B^t}_{ia}$ is the transpose of $\tilde{\Phi}^B_{ia}$ of Equation (91):

$$\begin{aligned} \tilde{\Phi}^{B^t}_{ia} &= \begin{bmatrix} \tilde{\Phi}^{B^t}_{ia1} & \tilde{\Phi}^{B^t}_{ia2} & \cdots & \tilde{\Phi}^{B^t}_{ian} \end{bmatrix} \\ &= - \begin{bmatrix} \tilde{\Phi}^B_{ia1} & \tilde{\Phi}^B_{ia2} & \cdots & \tilde{\Phi}^B_{ian} \end{bmatrix} \end{aligned} \tag{4-93}$$

Again, to form $\tilde{\Phi}^{B^t}_{ia}$ we first form $\tilde{\Phi}^B_{ia}$ as in Equation (91), then transpose this matrix, and while we transpose the matrix, we replace each element by its transpose. We do not form $\tilde{\Phi}^{B^t}_{ia}$ by first forming $\Phi^{B^t}_{ia}$ and then replacing each vector elements by its symmetric dyadic; in fact, as shown in Equation (93), we actually want the negative of this. It is easy to see that

$$\dot{\xi}_a^{B^t} \tilde{\Phi}_{ia}^B = - \tilde{\Phi}_{ia}^{B^t} \dot{\xi}_a^B \qquad (4\text{-}94)$$

Comparing Equations (90), (92), and (94) *now shows that this* peculiar minus sign is just what is required to get $\dot{\tilde{R}}_{ia} = -\dot{\tilde{R}}_{ia}^t$.

Since Φ_{ia}^B is a column matrix of vectors, we get, by a simple generalization of Equation(2-57),the result

$$\dot{\Phi}_a^B = {}^B\dot{\Phi}_{ia}^B + \tilde{\omega}^B \cdot \Phi_{ia}^B \qquad (4\text{-}95)$$

where ${}^B\dot{\Phi}_{ia}^B$ is the time derivative of Φ_{ia}^B with respect to frame B:

$$ {}^B\dot{\Phi}_{ia}^B = \sum_k \dot{\xi}_{ak}^B \frac{\partial \Phi_{ia}^B}{\partial \xi_{ak}^B} \qquad (4\text{-}96)$$

It is now convenient to introduce the operator $\nabla_{\xi_a^B}$ as follows

$$\nabla_{\xi_a^B} = \begin{bmatrix} \dfrac{\partial}{\partial \xi_{a1}^B} \\ \\ \dfrac{\partial}{\partial \xi_{a2}^B} \\ \\ \vdots \\ \\ \dfrac{\partial}{\partial \xi_{an}^B} \end{bmatrix} \qquad (4\text{-}97)$$

If we apply $\nabla_{\xi_a^B}$ to the vector $\vec{R}_{ia}$ we get a column matrix of vectors:

$$\Phi_{ia}^{B} = \nabla_{\xi_a^B} \vec{R}_{ia} \tag{4-98}$$

where we have made use of Equations (3-3) and (3-5). If we apply $\nabla_{\xi_a^B}$ to a row matrix of vectors we get

$$\nabla_{\xi_a^B} \Phi_{ia}^{B^t} = \nabla_{\xi_a^B} \begin{bmatrix} \vec{\Phi}_{ia1}^{B} & \vec{\Phi}_{ia2}^{B} & \cdots & \vec{\Phi}_{ian}^{B} \end{bmatrix} \tag{4-99}$$

$$= \begin{bmatrix} \nabla_{\xi_a^B} \vec{\Phi}_{ia1}^{B} & \nabla_{\xi_a^B} \vec{\Phi}_{ia2}^{B} & \cdots & \nabla_{\xi_a^B} \vec{\Phi}_{ian}^{B} \end{bmatrix}$$

Note that each of the elements $\nabla_{\xi_a^B} \vec{\Phi}_{iak}^{B}$ is actually a column matrix of vectors, and hence $\nabla_{\xi_a^B} \Phi_{ia}^{B^t}$ is an $n \times n$ matrix of vectors. The transpose of this matrix is

$$\left(\nabla_{\xi_a^B} \Phi_{ia}^{B^t}\right)^t = \begin{bmatrix} \left(\nabla_{\xi_a^B} \vec{\Phi}_{ia1}^{B}\right)^t \\ \left(\nabla_{\xi_a^B} \vec{\Phi}_{ia2}^{B}\right)^t \\ \vdots \\ \left(\nabla_{\xi_a^B} \vec{\Phi}_{ian}^{B}\right)^t \end{bmatrix} \tag{4-100}$$

We now return to Equation (96) and take its transpose:

$$\dot{\Phi}^{B^t}_{ia} = \sum_k \dot{\xi}^B_{ak} \frac{\partial \Phi^{B^t}_{ia}}{\partial \xi^B_{ak}} = \dot{\xi}^{B^t}_a \nabla_{\xi^B_a} \Phi^{B^t}_{ia} \tag{4-101}$$

Taking the transpose of this and substituting into Equation (95) yields

$$\dot{\Phi}^B_a = \left(\nabla_{\xi^B_a} \Phi^{B^t}_{ia} \right)^t \dot{\xi}^B_a + \tilde{\omega}^B \cdot \Phi^B_{ia} \tag{4-102}$$

<u>Evaluation of $\bar{D}$</u>

From Equations (90) and (102) we get immediately

$$M\dot{\tilde{R}}_{ca} = \sum_i m_i \dot{\tilde{R}}_{ia} = \dot{\xi}^{B^t}_a \tilde{\alpha}^B_a + \tilde{\omega}^B \cdot M\tilde{R}_{ca} - M\tilde{R}_{ca} \cdot \tilde{\omega}^B \tag{4-103}$$

$$\dot{\alpha}^B_a = \sum_i m_i \dot{\Phi}^B_{ia} = \left(\nabla_{\xi^B_a} \alpha^{B^t}_a \right)^t \dot{\xi}^B_a + \tilde{\omega}^B \cdot \alpha^B_a \tag{4-104}$$

where we have made use of Equation(3-17)in the form

$$\tilde{\alpha}^B_a = \sum_i m_i \tilde{\Phi}^B_{ia} \tag{4-105}$$

$$\nabla_{\xi^B_a} \alpha^{B^t}_a = \sum_i m_i \nabla_{\xi^B_a} \Phi^{B^t}_{ia} \tag{4-106}$$

When substituting Equation (90) for $\dot{\tilde{R}}_{ia}$ into Equation (69)

for $\bar{D}_{22}$, we note that in the third term $\tilde{\omega}^B$ gets "caught" between $\tilde{R}_{ia}$ on the left and $\tilde{R}_{ia}^t$ on the right. In order to pull $\vec{\omega}^B$ out of the summation over the particles, we make use of the following*:

$$\tilde{R}_{ia} \cdot \tilde{\omega}^B \cdot \tilde{R}_{ia}^t = \vec{\omega}^B \cdot \vec{R}_{ia}\, \tilde{R}_{ia} = \tilde{R}_{ia}\, \vec{R}_{ia} \cdot \vec{\omega}^B \tag{4-107}$$

Note that $\vec{R}_{ia}\, \tilde{R}_{ia}$ and $\tilde{R}_{ia}\, \vec{R}_{ia}$ are triadics. We run into a similiar situation when we substitute $\tilde{\dot{R}}_{ia}$ into Equation (71) for $\bar{D}_{23}$. In this case, $\tilde{\omega}^B$ gets caught between $\tilde{R}_{ia}$ on the left and $\Phi_{ia}^{B^t}$ on the right. In this case, we use

$$\tilde{\omega}^B \cdot \Phi_{ia}^{B^t} = \tilde{\Phi}_{ia}^{B^t} \cdot \vec{\omega}^B \tag{4-108}$$

When substituting Equation (102) for $\dot{\Phi}_a^B$ into Equations (70) and (72) to get $\bar{D}_{32}$ and $\bar{D}_{33}$, respectively, we make use of

$$\tilde{\omega}^B \cdot \Phi_{ia}^B = \vec{\omega}^B \cdot \tilde{\Phi}_{ia}^B \tag{4-109}$$

The final result is that $\bar{D}$ is given as follows*:

$$\bar{D} = \begin{bmatrix} \ddot{0} & \ddot{0} & 0 \\ \left\{\dot{\xi}_a^{B^t} \gamma_a^B + \tilde{\omega}^B \cdot M\tilde{R}_{ca} - M\tilde{R}_{ca} \cdot \tilde{\omega}^B\right\} & \left\{\dot{\xi}_a^{B^t} \sum_i m_i \Phi_{ia}^B \cdot \tilde{R}_{ia}^t + \tilde{\omega}^B \cdot \Upsilon_a - \vec{\omega}^B \cdot \sum_i m_i \vec{R}_{ia} \tilde{R}_{ia}\right\} & \left\{\dot{\xi}_a^{B^t} \sum_i m_i \Phi_{ia}^B \cdot \Phi_{ia}^{B^t} + \sum_i m_i \Phi_{ia}^{B^t} \cdot \tilde{R}_{ia}^t \cdot \vec{\omega}^B\right\} \\ \left\{\left(\nabla_{\xi_a^B} \gamma_a^{B^t}\right)^t \dot{\xi}_a^B + \tilde{\omega}^B \cdot \gamma_a^B\right\} & \left\{\sum_i m_i R_{ia} \cdot \left(\nabla_{\xi_a^B} \Phi_{ia}^{B^t}\right)^t \dot{\xi}_a^B + \vec{\omega}^B \cdot \sum_i m_i \tilde{\Phi}_{ia}^B \cdot \tilde{R}_{ia}^t\right\} & \left\{\sum_i m_i \left[\left(\nabla_{\xi_a^B} \Phi_{ia}^{B^t}\right)^t \dot{\xi}_a^B\right] \cdot \Phi_{ia}^{B^t} + \vec{\omega}^B \cdot \sum_i m_i \tilde{\Phi}_{ia}^B \cdot \Phi_{ia}^{B^t}\right\} \end{bmatrix} \tag{4-110}$$

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* See Appendix C

Evaluation of $\overline{X}$

We can now get explicit expressions for $\overline{X}$ by using this $\overline{D}$ in Equation (78). We first notice that the first element of $\overline{X}$, namely $\overline{X}_{\vec{v}_a}$ is zero. The second element of $\overline{X}$, namely $\overline{X}_{\vec{\omega}^B}$, was simplified as shown in Equation (84) because of Equation (81). We can explicitly verify Equation (81) by use of $\overline{D}$ of Equation (110). When we do this, we must make use of the following*

$$\tilde{\omega}^B \cdot \overleftrightarrow{I}_a \cdot \vec{\omega}^B = \vec{\omega}^B \cdot \sum_i m_i \vec{R}_{ia} \vec{R}_{ia} \cdot \tilde{\omega}^B \tag{4-111}$$

$$= \vec{\omega}^B \cdot \sum_i m_i \vec{R}_{ia} \tilde{R}_{ia} \cdot \vec{\omega}^B$$

and

$$\dot{\xi}_a^{B^t} \sum_i m_i \tilde{\Phi}_{ia}^B \cdot \Phi_{ia}^{B^t} \dot{\xi}_a^B = \vec{0} \tag{4-112}$$

Equation (111) follows from writing $\overleftrightarrow{I}_a$ in the form

$$\overleftrightarrow{I}_a = \sum_i m_i \left[(\vec{R}_{ia} \cdot \vec{R}_{ia}) \overleftrightarrow{E} - \vec{R}_{ia} \vec{R}_{ia} \right] \tag{4-113}$$

and Equation (112) follows because $\tilde{\Phi}_{ia}^B \cdot \Phi_{ia}^{B^t}$ is an $n \times n$ skew-symmetric matrix of vectors. It then follows that

$$\overline{X}_{\vec{\omega}^B} = -M\dot{\tilde{R}}_{ca} \cdot \vec{v}_a = \vec{v}_a \times \vec{P} \tag{4-114}$$

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* See Appendix C

Rather than writing out all of $\bar{X}_{\dot{\xi}^B_a}$, it is convenient to just write out the j^{th} element, $\bar{X}_{\dot{\xi}^B_{aj}}$. This j^{th} element is obtained by just using the j^{th} element in the column matrices $\dot{\alpha}^B_a$ and $\bar{D}_{32}$, and using just the j^{th} row of the matrix $\bar{D}_{33}$. Thus, from Equation (78) or (84) we have

$$\bar{X}_{\dot{\xi}^B_{aj}} = -\dot{\vec{\alpha}}^B_{aj} \cdot \vec{v}_a - (\bar{D}_{32})_j \cdot \vec{\omega}^B - (\bar{D}_{33})_{j.} \dot{\xi}^B_a \tag{4-115}$$

where

$$\dot{\vec{\alpha}}^B_{aj} = \sum_i m_i \dot{\vec{\Phi}}^B_{iaj} \tag{4-116}$$

$$(\bar{D}_{32})_j = \sum_i m_i \dot{\vec{\Phi}}^B_{iaj} \cdot \tilde{R}^t_{ia} \tag{4-117}$$

$$(\bar{D}_{33})_{j.} = \sum_i m_i \dot{\vec{\Phi}}^B_{iaj} \cdot \Phi^{B^t}_{ia} \tag{4-118}$$

When we take the appropriate rows out of $\bar{D}$ as given in Equation (110) we find

$$\bar{X}_{\dot{\xi}^B_{aj}} = -\left(\nabla_{\xi^B_a} \vec{\alpha}^B_{aj}\right)^t \dot{\xi}^B_a \cdot \vec{v}_a - \tilde{\omega}^B \cdot \vec{\alpha}^B_{aj} \cdot \vec{v}_a \tag{4-119}$$

$$- \sum_i m_i \tilde{R}_{ia} \cdot \left(\nabla_{\xi^B_a} \vec{\Phi}^B_{iaj}\right)^t \dot{\xi}^B_a \cdot \vec{\omega}^B - \vec{\omega}^B \cdot \sum_i m_i \tilde{\Phi}^B_{iaj} \cdot \tilde{R}^t_{ia} \cdot \vec{\omega}^B$$

$$- \dot{\xi}^{B^t}_a \sum_i m_i \left(\nabla_{\xi^B_a} \vec{\Phi}^B_{iaj}\right) \cdot \Phi^{B^t}_{ia} \dot{\xi}^B_a - \vec{\omega}^B \cdot \sum_i m_i \tilde{\Phi}^B_{iaj} \cdot \Phi^{B^t}_{ia} \dot{\xi}^B_a$$

Now notice that the term which is quadratic in $\vec{\omega}^B$ can be written as

$$-\vec{\omega}^B \cdot \sum_i m_i \tilde{\Phi}^B_{iaj} \cdot \tilde{R}^t_{ia} \cdot \vec{\omega}^B = -\frac{1}{2}\vec{\omega}^B \cdot \sum_i m_i \left[\tilde{\Phi}^B_{iaj} \cdot \tilde{R}^t_{ia} + \tilde{R}_{ia} \cdot \tilde{\Phi}^{B^t}_{iaj}\right] \cdot \vec{\omega}^B \tag{4-120}$$

and similiarly for the term which is quadratic in $\dot{\xi}^B_a$:

$$-\dot{\xi}^{B^t}_a \sum_i m_i \left(\nabla_{\xi^B_a} \vec{\Phi}^B_{iaj}\right) \cdot \Phi^{B^t}_{ia} \dot{\xi}^B_a = -\frac{1}{2}\dot{\xi}^{B^t}_a \sum_i m_i \left[\left(\nabla_{\xi^B_a} \vec{\Phi}^B_{iaj}\right) \cdot \Phi^{B^t}_{ia} + \Phi^B_{ia} \cdot \left(\nabla_{\xi^B_a} \vec{\Phi}^B_{iaj}\right)^t\right] \dot{\xi}^B_a \tag{4-121}$$

In both Equations (120) and (121), we replaced a term by its symmetric part because the skew-symmetric part drops out of the quadratic form expression.

Next we make use of Equations(3-3) and (98) and write

$$\nabla_{\xi^B_a} \vec{\Phi}^B_{iaj} = \nabla_{\xi^B_a}\left(\frac{\partial \vec{R}_{ia}}{\partial \xi^B_{aj}}\right) = \frac{\partial}{\partial \xi^B_{aj}}\left(\nabla_{\xi^B_a} \vec{R}_{ia}\right) = \frac{\partial \Phi^B_{ia}}{\partial \xi^B_{aj}} \tag{4-122}$$

where we have assumed that $\vec{R}_{ia}$ and its first two partial derivatives are continuous so that the order of partial differentiation can be interchanged. Similiarly we get

$$\vec{\alpha}^B_{aj} = \sum_i m_i \vec{\Phi}^B_{iaj} = \frac{\partial}{\partial \xi^B_{aj}}\left(\sum_i m_i \vec{R}_{ia}\right) = \frac{\partial}{\partial \xi^B_{aj}}(M\vec{R}_{ca}) \tag{4-123}$$

$$\nabla_{\xi^B_a} \vec{\alpha}^B_{aj} = \frac{\partial}{\partial \xi^B_{aj}} \left(\sum_i m_i \nabla_{\xi^B_a} \vec{R}_{ia} \right) = \frac{\partial}{\partial \xi^B_{aj}} \left(\sum_i m_i \Phi^B_{ia} \right) = \frac{\partial \alpha^B_a}{\partial \xi^B_{aj}} \tag{4-124}$$

Making use of Equations (120) to (124), $\overline{X}_{\dot{\xi}^B_{aj}}$ now becomes

$$\begin{aligned} \overline{X}_{\dot{\xi}^B_{aj}} &= -\vec{v}_a \cdot M \left(\frac{\partial \widetilde{R}^t_{ca}}{\partial \xi^B_{aj}} \right) \cdot \vec{\omega}^B - \vec{v}_a \cdot \left(\frac{\partial \alpha^{B^t}_a}{\partial \xi^B_{aj}} \right) \dot{\xi}^B_a - \frac{1}{2} \vec{\omega}^B \cdot \left(\frac{\partial \overleftrightarrow{I}_a}{\partial \xi^B_{aj}} \right) \cdot \vec{\omega}^B \\ &\quad - \vec{\omega}^B \cdot \left(\frac{\partial \beta^{B^t}_a}{\partial \xi^B_{aj}} \right) \dot{\xi}^B_a - \frac{1}{2} \dot{\xi}^{B^t}_a \left(\frac{\partial \gamma^B_a}{\partial \xi^B_{aj}} \right) \dot{\xi}^B_a \qquad (4\text{-}125) \\ &= - \frac{\partial T}{\partial \xi^B_{aj}} \end{aligned}$$

where T is given in Equation(3-26) or(3-29). Thus, we have now established that

$$\overline{X}_{\dot{\xi}^B_a} = - \nabla_{\xi^B_a} T \tag{4-126}$$

as already indicated in Equation (88).

Evaluation of $\overline{Y}$

To get $\overline{Y}$ we form the transpose of $\overline{D}$ of Equation (110) and dot multiply it into $\overline{\sigma}$. Since the first column of $\overline{D}^t$ is zero, and none of the elements of $\overline{D}$ involve $\vec{v}_a$, it follows that $\overline{Y}$ does not involve $\vec{v}_a$. Now each element of the second column of $\overline{D}^t$ has a term which ends in $\widetilde{\omega}^B$, and this term drops out when we multiply it into $\widetilde{\omega}^B$; therefore, $\overline{Y}$ does not involve all of $\overline{D}^t$ just as we found earlier that $\overline{X}$ does not involve all of $\overline{D}$.

When we multiply $\overline{D}^t$ into $\overline{\sigma}$, the result is

$$\overline{Y}_{\vec{v}_a} = \widetilde{\omega}^B \cdot M\widetilde{R}^t_{ca} \cdot \vec{\omega}^B + 2\widetilde{\omega}^B \cdot \alpha^{B^t}_a \dot{\xi}^B_a + \dot{\xi}^{B^t}_a \left(\nabla_{\xi^B_a} \alpha^{B^t}_a \right) \dot{\xi}^B_a \tag{4-127}$$

$$\overline{Y}_{\vec{\omega}^B} = \widetilde{\omega}^B \cdot \overleftrightarrow{I}_a \cdot \vec{\omega}^B + 2\dot{\xi}^{B^t}_a \sum_i m_i \widetilde{R}^t_{ia} \cdot \widetilde{\Phi}^B_{ia} \cdot \vec{\omega}^B + \dot{\xi}^{B^t}_a \sum_i m_i \widetilde{R}_{ia} \cdot \left(\nabla_{\xi^B_a} \Phi^{B^t}_{ia} \right) \dot{\xi}^B_a \tag{4-128}$$

$$\overline{Y}_{\dot{\xi}^B_a} = \vec{\omega}^B \cdot \sum_i m_i \widetilde{\Phi}^B_{ia} \cdot \widetilde{R}_{ia} \cdot \vec{\omega}^B + 2\vec{\omega}^B \cdot \sum_i m_i \Phi^B_{ia} \cdot \widetilde{\Phi}^{B^t}_{ia} \dot{\xi}^B_a \tag{4-129}$$
$$+ \sum_i m_i \Phi^B_{ia} \cdot \left(\dot{\xi}^{B^t}_a \nabla_{\xi^B_a} \Phi^{B^t}_{ia} \right) \dot{\xi}^B_a$$

In order to separate $\dot{\xi}^B_a$ from the particle dependent factors in $\overline{Y}_{\dot{\xi}^B_a}$, we write out just the j^{th} term as follows:

$$\overline{Y}_{\dot{\xi}^B_{aj}} = \vec{\omega}^B \cdot \sum_i m_i \widetilde{\Phi}^B_{iaj} \cdot \widetilde{R}_{ia} \cdot \vec{\omega}^B + 2\vec{\omega}^B \cdot \sum_i m_i \vec{\Phi}^B_{iaj} \cdot \widetilde{\Phi}^{B^t}_{ia} \dot{\xi}^B_a \tag{4-130}$$
$$+ \dot{\xi}^{B^t}_a \sum_i m_i \vec{\Phi}^B_{iaj} \cdot \left(\nabla_{\xi^B_a} \Phi^{B^t}_{ia} \right) \dot{\xi}^B_a$$

V. SUMMARY OF DYNAMICS EQUATIONS FOR A DEFORMABLE BODY

In the previous section, we have used the transformation operator formalism to generate a set of momentum formulation equations plus a set of velocity formulation equations. The momentum formulation equations are the following:

$$\begin{bmatrix} \dot{\vec{P}} \\ \dot{\vec{H}}_a \\ \dot{g}_a^B \end{bmatrix} + \begin{bmatrix} \vec{0} \\ \vec{v}_a \times \vec{P} \\ \bar{X}_{\dot{\xi}_a^B} \end{bmatrix} = \begin{bmatrix} \vec{F} \\ \vec{L}_a \\ k_a^B \end{bmatrix} \tag{5-1}$$

and

$$\begin{bmatrix} \vec{P} \\ \vec{H}_a \\ g_a^B \end{bmatrix} = \begin{bmatrix} M\overleftrightarrow{E} & M\tilde{R}_{ca}^t & \alpha_a^{B^t} \\ M\tilde{R}_{ca} & \overleftrightarrow{I}_a & \beta_a^{B^t} \\ \alpha_a^B & \beta_a^B & \gamma_a^B \end{bmatrix} \cdot \begin{bmatrix} \vec{v}_a \\ \vec{\omega}^B \\ \dot{\xi}_a^B \end{bmatrix} \tag{5-2}$$

The velocity formulation equations are

$$\begin{bmatrix} M\overleftrightarrow{E} & M\tilde{R}_{ca}^t & \alpha_a^{B^t} \\ M\tilde{R}_{ca} & \overleftrightarrow{I}_a & \beta_a^{B^t} \\ \alpha_a^B & \beta_a^B & \gamma_a^B \end{bmatrix} \cdot \begin{bmatrix} \dot{\vec{v}}_a \\ \dot{\vec{\omega}}^B \\ \ddot{\xi}_a^B \end{bmatrix} + \begin{bmatrix} \bar{Y}_{\vec{v}_a} \\ \bar{Y}_{\vec{\omega}^B} \\ \bar{Y}_{\dot{\xi}_a^B} \end{bmatrix} = \begin{bmatrix} \vec{F} \\ \vec{L}_a \\ k_a^B \end{bmatrix} \tag{5-3}$$

These dynamics equations form a complete set (in fact, two complete sets) because they allow for the determination (say, by numerical integration) of the 6 external and n internal degree of freedom motion. Of course, we must also add some kinematic equations, but these equations are straightforward.

The dynamics equations are completely general because we have not introduced any constitutive equations to characterize a particular material. The method of specializing these equations is to specify Φ_{ia}^{B} for i = 1 to N. If we specify Φ_{ia}^{B} via the eigenvectors of the linearized dynamics equations, then the ξ_{a}^{B} are the modal coordinates, and we get the dynamic equations of Bodley and Park[28,38]; if we linearize our dynamics equations about zero velocity, we get the linearized vibration equations.[87] In such a case, we would also express k_{a}^{BI} (see Equations (4-56) to (4-58)) in terms of a stiffness matrix and perhaps in terms of a damping matrix. In this case, Φ_{ia}^{B} could be considered as constant with respect to frame B (which has inertial angular velocity $\vec{\omega}^{B}$), and hence, all the partial derivatives of Φ_{ia}^{B} would be zero.

Our equations are really much more useful than to just use them in a linearized form. As a matter of fact, the equations can easily be used in their full generality in the case of a collection of rigid bodies. In this case, the internal coordinates ξ_{a}^{B} can be used to represent the relative coordinates (rotational and/or translational) between the rigid bodies; frame B can be fixed in one of the bodies, say in the main body; k_{a}^{BI} now represents interbody torques and/or forces due to actuators (interbody torques and forces which are due to constraints drop out). The resulting equations are given in Reference (69).

We can, of course, also apply our equations to a collection of flexible bodies by introducing appropriate Φ_{ia}^{B} and k_{a}^{BI}. This

subject will be addressed in a future report.

All the dynamics equations of References (1) to (54) have the general form of Equations (5-1) and (5-2), or of Equation (5-3). Evidently, all of these equations are straightforward consequences of Newton's law for a particle: all we need to do is to linearly transform from particle velocities, momenta, and forces to system velocities, momenta, and forces, respectively. In order to obtain the extra term $\overline{X}$ or $\overline{Y}$, we need to evaluate the time derivative of the transformation operator A; but this too is an entirely algebraic process.

The equivalence of various forms of velocity equations has been pointed out by Likins[88]. In the present work we have extended this equivalence to the momentum equations and velocity equations for an arbitrary deformable body.

VI. CONCLUDING COMMENTS

We have seen that it is possible to obtain the exact dynamics equations for a deformable body by using purely vectorial techniques. The result is the same as that obtainable from Lagrange's or Hamilton's equations, but we do not have to perform partial derivatives of the kinetic energy (except, as we did, to show that the results are equivalent). Constraint forces (and torques) drop out in our formulation for precisely the same reason that they drop out of Lagrange's equation. In fact, we used essentially the same procedure to derive our equations as Lagrange used to derive Lagrange's equations.

Our approach is very similiar to that of Kron and that used in "matrix structural analysis", and in the "finite element" method. However, there are also some differences between our method and that of others. A major difference is that we use classical vectorial mechanics to get all the results we need. We introduce angular velocity vectors (rather than limiting ourselves to time derivatives of Euler angles); in this respect, our equations are similiar to those obtained by Boltzmann and Hamel, sometimes called Lagrange equations in quasi-coordinates.[80] Such nonholonomic velocities are also used extensively by Kane.

Finally, we should point out that we have developed both a set of momentum equations and a set of velocity equations. One set may be more useful than the other set, depending on the particular problem at hand. Unfortunately, momentum formulations are unfamiliar to many and are not used much. Russell, and Vance and Sitchin are conspicuous exceptions.

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NOMENCLATURE

Symbol	Definition	Page (equation) of first occurrence
A	transformation operator which transforms $\overline{\sigma}$ to σ	35(4-12) and 43(4-52)
B	left inverse of transformation operator A (i.e., transformation operator which transforms σ to $\overline{\sigma}$)	37(4-25)
$\overline{C}$	coefficient of $\overline{\mu}$ in the expansion of $\dot{\overline{\mu}}$	38(4-30)
$\overline{D}$	equals $\overline{C} \cdot \overline{\mu}$	39(4-33) and 48(4-67)
$\overline{D}_{ij}$	(i,j) -- element of $\overline{D}$, for i,j = 2,3	48(4-68)
$(\overline{D}_{32})_j$	j^{th} element of column matrix $\overline{D}_{32}$	60(4-115)
$(\overline{D}_{33})_{j\cdot}$	j^{th} row of matrix $\overline{D}_{33}$	60(4-115)
$\overleftrightarrow{E}$	identity dyadic	24(2-73)
$\vec{f}_i$	force on particle i	12(2-14)
$\vec{f}_i^E$	external force on particle i	12(2-19)
$\vec{f}_i^I$	internal force on particle i	12(2-18)
$\vec{F}$	total force on system of N particles	12(2-17)
$\vec{F}^E$	external force on system of N particles	13(2-19)
g_a^B	internal generalized momentum, relative to point a and frame B	31(3-33)

NOMENCLATURE - Continued

Symbol	Definition	Page (equation) of first occurrence
g_c^B	special case of g_a^B, when a is the center of mass c	47(4-62)
G	primitive system momentum; column matrix of N vectors, with $\vec{p}_i$ as the i^{th} element	41(4-40)
$\overline{G}$	transformed system momentum; column matrix whose elements are $\vec{P}$, $\vec{H}_a$, and g_a^B	35(4-13) and 44(4-54)
$\vec{h}_{ai}$	angular momentum, about point a, of particle i	17(2-38)
$\vec{H}_a$	total angular momentum, about point a, of system of N particles	17(2-39)
$\vec{H}_c$	total angular momentum, about center of mass, of system of N particles	18(2-41)
$\overleftrightarrow{I}_a$	total inertia dyadic, about point a, for system of N particles	22(2-66)
$\overleftrightarrow{I}_c$	total inertia dyadic, about center of mass, for system of N particles	23(2-67)
k_a^B	generalized force, relative to point a and frame B, on internal degrees of freedom	29(3-22)

NOMENCLATURE - Continued

Symbol	Definition	Page (equation) of first occurrence
$k_a^{B^t}$	transpose of k_a^B	29(3-23)
k_a^{BE}	external part of k_a^B	45(4-56)
k_a^{BI}	internal part of k_a^B	45(4-56)
K	primitive system force; column matrix of N vectors, with $\vec{f}_i$ as the i^{th} element	33(4-1) and 41(4-41)
$\overline{K}$	transformed system force; column matrix whose elements are $\vec{F}$, $\vec{L}_a$, and k_a^B	35(4-14) and 45(4-55)
$\vec{\ell}_{ai}$	moment of force, about point a, on particle i	20(2-51)
$\vec{L}_a$	total moment or torque, about point a, on system of N particles	19(2-46)
$\vec{L}_c$	total moment or torque, about center of mass, on system of N particles	19(2-50)
$\vec{L}_a^E$	total external moment or torque, about point a, on system of N particles	19(2-46)
m_i	mass of particle i	9(2-2)
M	total mass of system of N particles	9(2-3)

NOMENCLATURE - Continued

Symbol	Definition	Page (equation) of first occurrence
n	number of internal degrees of freedom of system of N particles	25
N	number of particles in deformable body	9
$\overleftrightarrow{O}$	zero dyadic	48(4-67)
$\vec{p}_i$	linear momentum of particle i	10(2-5)
$\vec{P}$	total linear momentum of system of N particles	10(2-6)
$\vec{r}_a$	position vector to point a from inertial reference origin	14(2-23)
$\vec{r}_c$	position vector to center of mass	9(2-2)
$\vec{r}_i$	position vector to particle i from inertial reference origin	9(2-1)
$\dot{\vec{r}}_i$	inertial time derivative of $\vec{r}_i$	9(2-1)
$\vec{R}_{ca}$	position vector to center of mass from point a	14(2-24)
$\widetilde{R}_{ca}$	skew - symmetric dyadic formed from vector $\vec{R}_{ca}$	22(2-63)
$\widetilde{R}^t_{ca}$	dyadic transpose of $\widetilde{R}_{ca}$	22(2-62)
$\vec{R}_{ia}$	position vector to particle i from point a	14(2-23)
$\dot{\vec{R}}_{ia}$	inertial time derivative of $\vec{R}_{ia}$	14(2-26)

NOMENCLATURE - Continued

Symbol	Definition	Page (equation) of first occurrence
$\dot{\vec{R}}_{ia}^{\,2}$	$\vec{R}_{ia} \cdot \vec{R}_{ia}$	16(2-33)
$\tilde{R}_{ia}$	skew -- symmetric dyadic formed from vector $\vec{R}_{ia}$	22
$\tilde{R}_{ia}^{t}$	dyadic transpose of $\tilde{R}_{ia}$	21(2-60)
$\overset{B}{\vec{R}}_{ia}$	time derivative of $\vec{R}_{ia}$ with respect to frame B which has inertial angular velocity $\vec{\omega}^{B}$	21(2-58)
$\vec{R}_{ic}$	*position vector to particle i from center of mass*	14(2-27)
T_i	kinetic energy of particle i	10(2-8)
T	total kinetic energy of system of N particles	10(2-9)
T_a^*	kinetic energy function in terms of $\vec{v}_a$	17
T_c^*	kinetic energy function in terms of $\vec{v}_c$	17
$T_a^{B^*}$	kinetic energy function in terms of $\vec{v}_a$ and $\vec{\omega}^{B}$	23
$\vec{u}_{ia}^{B}$	alternate notation for $\overset{B}{\vec{R}}_{ia}$	21(2-59)
$\vec{u}_{ia}^{B\,2}$	$\vec{u}_{ia}^{B} \cdot \vec{u}_{ia}^{B}$	22(2-64)

Symbol	Definition	Page (equation) of first occurrence
$\vec{v}_a$	velocity of point a	14(2-26)
$\vec{v}_a^2$	$\vec{v}_a \cdot \vec{v}_a$	16(2-33)
$\vec{v}_c$	velocity of center of mass	10(2-4)
$\vec{v}_i$	velocity of particle i	9(2-1)
$\dot{\vec{V}}$	inertial time derivative of vector $\vec{V}$	21(2-57)
$\overset{B}{\vec{V}}$	time derivative of vector $\vec{V}$ with respect to frame B which has inertial angular velocity $\vec{\omega}^B$	21(2-57)
X	a primitive term which is quadratic in σ or in G	33(4-1)
$\overline{X}$	a transformed term which is quadratic in $\overline{\sigma}$ or $\overline{G}$	36(4-18) and 50(4-78)
$\overline{X}_{\vec{v}_a}$	first part of the three-part column-decomposition of $\overline{X}$	51(4-84)
$\overline{X}_{\vec{\omega}^B}$	second part of the three-part column-decomposition of $\overline{X}$	51(4-84)
$\overline{X}_{\xi_a^B}$	third part of the three-part column-decomposition of $\overline{X}$	51(4-84)

Symbol	Definition	Page (equation) of first occurrence
$\overline{X}_{\dot{\xi}^B_{aj}}$	j^{th} element of $\overline{X}_{\dot{\xi}^B_a}$	51(4-115)
Y	a primitive term which is quadratic in σ or in G	34(4-3)
$\overline{Y}$	a transformed term which is quadratic in $\overline{\sigma}$ and $\overline{G}$	36(4-21) and 50(4-79)
$\overline{Y}_{\vec{v}_a}$	first part of the three-part column-decomposition of $\overline{Y}$	52(4-86)
$\overline{Y}_{\vec{\omega}^B}$	second part of the three-part column-decomposition of $\overline{Y}$	52(4-86)
$\overline{Y}_{\dot{\xi}^B_a}$	third part of the three-part column-decomposition of $\overline{Y}$	52(4-86)
$\overline{Y}_{\dot{\xi}^B_{aj}}$	j^{th} element of $\overline{Y}_{\dot{\xi}^B_a}$	52(4-130)
α^B_a	coefficient of $\vec{v}_a$ in expansion of internal generalized momentum g^B_a in terms of $\vec{v}_a$, $\vec{\omega}^B$, and $\dot{\xi}^B_a$	28(3-17) and 31(3-33)
$\alpha^{B^t}_a$	transpose of α^B_a	29(3-18)
α^B_c	special case of α^B_a, when a is the center of mass c	46(4-61)

Symbol	Definition	Page (equation) of first occurrence
β_a^B	coefficient of $\vec{\omega}^B$ in expansion of internal generalized momentum g_a^B in terms of $\vec{v}_a$, $\vec{\omega}^B$, and $\dot{\xi}_a^B$	29(3-19) and 31(3-33)
$\beta_a^{B^t}$	transpose of $\beta_a^{B^t}$	29(3-20)
β_c^B	special case of β_a^B, when a is the center of mass c	47(4-62)
γ_a^B	coefficient of $\dot{\xi}_a^B$ in expansion of internal generalized momentum g_a^B in terms of $\vec{v}_a$, $\vec{\omega}^B$, and $\dot{\xi}_a^B$	29(3-21) and 31(3-33)
γ_c^B	special case of γ_a^B, when a is the center of mass c	47(4-62)
μ	primitive system mass; diagonal N×N matrix with $m_i\overleftrightarrow{E}$ on the i^{th} diagonal	33(4-2) and 40(4-39)
$\overline{\mu}$	transformed system mass	36(4-15) and 46(4-59)
ν	inverse of μ	34(4-8)
$\overline{\nu}$	inverse of $\overline{\mu}$	38(4-28)
ξ_{aj}^B	j^{th} internal generalized coordinate relative to point a and frame B	25(3-1)
$\dot{\xi}_a^B$	column matrix of n scalars, with $\dot{\xi}_{aj}^B$ as j^{th} element	26(3-7)

Symbol	Definition	Page (equation) of first occurrence
$\dot{\xi}_a^{B^t}$	transpose of $\dot{\xi}_a^B$ (i.e., row matrix of n scalars, with $\dot{\xi}_{aj}^B$ as j^{th} element)	27(3-8)
$\dot{\xi}_c^B$	special case of $\dot{\xi}_a^B$, when a is the center of mass c	47(4-62)
σ	primitive system velocity; column matrix of N vectors, with $\vec{v}_i$ as the i^{th} element	33(4-2) and 39(4-38)
$\bar{\sigma}$	transformed system velocity; column matrix whose elements are $\vec{v}_a$, $\vec{\omega}^B$, and $\dot{\xi}_a^B$	35(4-12) and 42(4-47)
$\vec{\Phi}_{iaj}^B$	coefficients of expansion of $\vec{u}_{ia}^B$ in terms of $\dot{\xi}_{aj}^B$ (equal to partial derivative of $\vec{R}_{ia}$ with respect to ξ_{aj}^B)	25(3-1)
$\tilde{\Phi}_{iaj}^B$	skew-symmetric dyadic formed from vector $\vec{\Phi}_{iaj}^B$	54(4-91)
$\tilde{\Phi}_{iaj}^{B^t}$	dyadic transpose of $\tilde{\Phi}_{iaj}^B$	54(4-93)
Φ_{ia}^B	column matrix of n vectors, with $\vec{\Phi}_{iaj}^B$ as j^{th} element	26(3-5)
$\Phi_{ia}^{B^t}$	transpose of Φ_{ia}^B (i.e., row matrix of n vectors, with $\vec{\Phi}_{iaj}^B$ as j^{th} element)	26(3-6)
Φ_{ic}^B	special case of Φ_{ia}^B, when a is the center of mass c	47(4-63)

Symbol	Definition	Page (equation) of first occurrence
$\tilde{\Phi}^{B}_{ia}$	column matrix of n dyadics formed from Φ^{B}_{ia} by replacing all vector elements by the skew-symmetric dyadic of these vectors	53(4-91)
$\tilde{\Phi}^{B^t}_{ia}$	transpose of $\tilde{\Phi}^{B}_{ia}$	54(4-93)
${}^{B}\dot{\Phi}^{B}_{ia}$	time derivative of Φ^{B}_{ia} with respect to frame B	55(4-95)
$\vec{\omega}^{B}$	angular velocity of frame B with respect to inertial space	21(2-57)
$\tilde{\omega}^{B}$	skew-symmetric dyadic formed from vector $\vec{\omega}^{B}$	53(4-89)
$\nabla_{\xi^{B}_{a}}$	column matrix of n scalar partial differentiation operators, with $\frac{\partial}{\partial \xi^{B}_{aj}}$ as j^{th} element	55(4-97)
$\vec{0}$	zero vector	12(2-18)

APPENDIX

A. VECTORS, DYADICS AND TRIADICS

Our physical world includes objects which we like to count or measure. For this purpose the integers and real numbers were invented. Originally these numbers did not exist (only the counting and measuring existed); but eventually these numbers were accepted as existing mathematically because they could be used for counting and measuring. These numbers were used extensively in the geometry and physics of our 3-dimensional world. Many geometric and physical quantities actually have 3 real numbers associated with them, and therefore these geometric and physical quantities were eventually represented by 3-dimensional vectors. Originally, these vectors did not exist (only the 3 real numbers existed); but eventually these vectors were accepted as existing *mathematically because* they could be used for producing the 3 real numbers associated with the geometrical and physical quantities. Similiarly, dyadics eventually were accepted as existing because they could be used to generate vectors; and triadics were accepted as existing because they could be used to generate dyadics; etc.

Vectors

If $\hat{x}$, $\hat{y}$, and $\hat{z}$ are used to symbolically represent 3 orthogonal directions in our 3-dimensional space, then a vector $\vec{V}$ can be written as

$$\vec{V} = V_x \hat{x} + V_y \hat{y} + V_z \hat{z} \tag{A-1}$$

Note that we are not really adding V_x, V_y, and V_z; nor are we adding $\hat{x}$, $\hat{y}$, and $\hat{z}$. Equation (A-1) is merely a symbolic repre-

sentation of the fact that the geometric or physical quantity has the 3 real numbers V_x, V_y, and V_z in the 3 directions $\hat{x}$, $\hat{y}$, and $\hat{z}$, respectively. The operation of the dot product is then defined so that we can produce real numbers from vectors. Thus, for the 3 orthogonal directions $\hat{x}$, $\hat{y}$, $\hat{z}$ we define

$$\begin{array}{lll} \hat{x} \cdot \hat{x} = 1 & \hat{x} \cdot \hat{y} = 0 & \hat{x} \cdot \hat{z} = 0 \\ \hat{y} \cdot \hat{x} = 0 & \hat{y} \cdot \hat{y} = 1 & \hat{y} \cdot \hat{z} = 0 \\ \hat{z} \cdot \hat{x} = 0 & \hat{z} \cdot \hat{y} = 0 & \hat{z} \cdot \hat{z} = 1 \end{array} \tag{A-2}$$

Then we get the vector components V_x, V_y, V_z as follows

$$\begin{aligned} V_x &= \hat{x} \cdot \vec{V} = \vec{V} \cdot \hat{x} \\ V_y &= \hat{y} \cdot \vec{V} = \vec{V} \cdot \hat{y} \\ V_z &= \hat{z} \cdot \vec{V} = \vec{V} \cdot \hat{z} \end{aligned} \tag{A-3}$$

Vector components can be conveniently put into matrix form. Thus, we define V_Q and V_Q^t as follows

$$V_Q = \begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix} \tag{A-4}$$

$$V_Q^t = \begin{bmatrix} V_x & V_y & V_z \end{bmatrix}$$

We can similiarly put the 3 directions $\hat{x}$, $\hat{y}$, $\hat{z}$ in a matrix form. Thus, we define Q and Q^t as follows

$$Q = \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} \tag{A-5}$$

$$Q^t = [\hat{x} \quad \hat{y} \quad \hat{z}]$$

Using the ordinary rules for matrix multiplication, Equation (A-1) can now be written as

$$\vec{V} = V_Q^t\, Q = Q^t V_Q \tag{A-6}$$

Similiarly, Equation (A-2) can now be written as

$$Q \cdot Q^t = 1_3 \tag{A-7}$$

where 1_3 is the 3 × 3 unit matrix. Equation (A-3) can now be written as

$$V_Q = Q \cdot \vec{V} = \vec{V} \cdot Q \tag{A-8}$$

One might argue that matrices were not intended to be used with vectors as elements as in Equation (A-5). The only response to such an objection is that one then simply redefines the notion of a matrix, but uses the same old matrix symbols, just as in Equation (A-1) we tacitly redefined the notion of addition so that

it would also apply to vectors.

Dyadics

Physical quantities which have 9 real numbers associated with them are conveniently represented by a dyadic. Thus, the dyadic $\overleftrightarrow{D}$ can be written as

$$\begin{aligned} \overleftrightarrow{D} = \; & D_{xx}\hat{x}\hat{x} + D_{xy}\hat{x}\hat{y} + D_{xz}\hat{x}\hat{z} \\ & + D_{yx}\hat{y}\hat{x} + D_{yy}\hat{y}\hat{y} + D_{yz}\hat{y}\hat{z} \\ & + D_{zx}\hat{z}\hat{x} + D_{zy}\hat{z}\hat{y} + D_{zz}\hat{z}\hat{z} \end{aligned} \tag{A-9}$$

We denote the $\hat{x}$, $\hat{y}$, $\hat{z}$ components of the dyadic $\overleftrightarrow{D}$ by the matrix D_Q given by

$$D_Q = \begin{bmatrix} D_{xx} & D_{xy} & D_{xz} \\ D_{yx} & D_{yy} & D_{yz} \\ D_{zx} & D_{zy} & D_{zz} \end{bmatrix} \tag{A-10}$$

$\overleftrightarrow{D}$ can now be written as

$$\overleftrightarrow{D} = Q^t D_Q Q \tag{A-11}$$

Since $Q \cdot Q^t = 1_3$ we can solve this for D_Q as follows:

$$D_Q = Q \cdot \overleftrightarrow{D} \cdot Q^t \tag{A-12}$$

Thus, whereas it takes one dot multiplication with Q to produce a set of real numbers from a vector (see Equation (A-8)), it takes two dot multiplications with Q to produce a set of real numbers from a dyadic.

A dyadic times a vector is another vector. Thus,

$$\begin{aligned} \overleftrightarrow{D} \cdot \vec{A} &= (Q^t D_Q Q) \cdot (Q^t A_Q) \\ &= Q^t (D_Q A_Q) \\ &= Q^t B_Q \end{aligned} \tag{A-13}$$

Thus we can write

$$\overleftrightarrow{D} \cdot \vec{A} = \vec{B} \qquad \text{implies } D_Q A_Q = B_Q \tag{A-14}$$

Note that the side from which $\overleftrightarrow{D}$ is multiplied is important because

$$\begin{aligned} \vec{A} \cdot \overleftrightarrow{D} &= (A_Q^t\, Q) \cdot (Q^t D_Q Q) \\ &= (A_Q^t\, D_Q)\, Q \\ &= Q^t (D_Q^t\, A_Q) \\ &= Q^t\, C_Q \\ &= C_Q^t\, Q \end{aligned} \tag{A-15}$$

Thus we can write

$$\vec{A} \cdot \overleftrightarrow{D} = \vec{C} \qquad \text{implies } D_Q^t A_Q = C_Q \tag{A-16}$$

The transpose or conjugate of $\overleftrightarrow{D}$ is the dyadic $\overleftrightarrow{D}^t$ such that

$$\overleftrightarrow{D} \cdot \vec{A} = \vec{A} \cdot \overleftrightarrow{D}^t \tag{A-17}$$

We evidently have

$$\overleftrightarrow{D}^t = Q^t D_Q^t Q \tag{A-18}$$

Thus, the matrix of components of $\overleftrightarrow{D}^t$ is the transpose of the matrix of components of $\overleftrightarrow{D}$. If $\overleftrightarrow{D}$ equals $\overleftrightarrow{D}^t$ then $\overleftrightarrow{D}$ is called a symmetric dyadic; if $\overleftrightarrow{D}$ equals $-\overleftrightarrow{D}^t$ then $\overleftrightarrow{D}$ is called an anti-symmetric or skew-symmetric dyadic. Thus, $\overleftrightarrow{D}$ has the same symmetry as its matrix of components.

We can form a dyadic by juxtapositioning two vectors

$$\begin{aligned} \overleftrightarrow{C} = \vec{A}\vec{B} &= (Q^t A_Q)(B_Q^t Q) \\ &= Q^t (A_Q B_Q^t) Q \end{aligned} \tag{A-19}$$

Thus

$$\overleftrightarrow{C} = \vec{A}\vec{B} \qquad \text{implies } C_Q = A_Q B_Q^t \tag{A-20}$$

Note that in general, $\vec{A}\vec{B}$ is not the same dyadic as $\vec{B}\vec{A}$. Even though we can form a dyadic out of two vectors, not every dyadic can be formed this simply. In the general case, it takes three vectors to specify a dyadic. For example, the dyadic $\overleftrightarrow{D}$

of Equation (A-9) can be written as

$$\overleftrightarrow{D} = \hat{x}\vec{V}_x + \hat{y}\vec{V}_y + \hat{z}\vec{V}_z \tag{A-21}$$

where

$$\begin{aligned}
\vec{V}_x &= D_{xx}\hat{x} + D_{xy}\hat{y} + D_{xz}\hat{z} \\
\vec{V}_y &= D_{yx}\hat{x} + D_{yy}\hat{y} + D_{yz}\hat{z} \\
\vec{V}_z &= D_{zx}\hat{x} + D_{zy}\hat{y} + D_{zz}\hat{z}
\end{aligned} \tag{A-22}$$

Thus the 3 vectors $\vec{V}_x$ $\vec{V}_y$, and $\vec{V}_z$ (together with the basis vectors $\hat{x}$, $\hat{y}$, and $\hat{z}$) represent the dyadic $\overleftrightarrow{D}$. Alternatively, we can represent $\overleftrightarrow{D}$ by $\vec{W}_x$, $\vec{W}_y$, and $\vec{W}_z$ as follows:

$$\overleftrightarrow{D} = \vec{W}_x\hat{x} + \vec{W}_y\hat{y} + \vec{W}_z\hat{z} \tag{A-23}$$

where

$$\begin{aligned}
\vec{W}_x &= D_{xx}\hat{x} + D_{yx}\hat{y} + D_{zx}\hat{z} \\
\vec{W}_y &= D_{xy}\hat{x} + D_{yy}\hat{y} + D_{zy}\hat{z} \\
\vec{W}_z &= D_{xz}\hat{x} + D_{yz}\hat{y} + D_{zz}\hat{z}
\end{aligned} \tag{A-24}$$

For any vector $\vec{A}$ there is a dyadic $\widetilde{A}$ such that the vector cross product becomes a dyadic dot product. Thus for any $\vec{A}$ and $\vec{B}$:

$$\vec{A} \times \vec{B} = \widetilde{A} \cdot \vec{B} \tag{A-25}$$

Since $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$ we also have

$$\vec{A} \times \vec{B} = -\widetilde{B} \cdot \vec{A} \tag{A-26}$$

If A_x, A_y, A_z are the $\hat{x}$, $\hat{y}$, $\hat{z}$ components of $\vec{A}$, then $\widetilde{A}$ can be given in $\hat{x}$, $\hat{y}$, $\hat{z}$ components as follows

$$\begin{aligned} \widetilde{A} = & -A_z \hat{x}\hat{y} + A_y \hat{x}\hat{z} + A_z \hat{y}\hat{x} \\ & -A_x \hat{y}\hat{z} - A_y \hat{z}\hat{x} + A_x \hat{z}\hat{y} \end{aligned} \tag{A-27}$$

Thus, we can write

$$\widetilde{A} = Q^t \widetilde{A}_Q Q \tag{A-28}$$

where

$$\widetilde{A}_Q = \begin{bmatrix} 0 & -A_z & A_y \\ A_z & 0 & -A_x \\ -A_y & A_x & 0 \end{bmatrix} \tag{A-29}$$

Since $\widetilde{A}_Q$ is skew-symmetric, so is $\widetilde{A}$:

$$\widetilde{A}^t = -\widetilde{A} \tag{A-30}$$

Using Equations (A-17) and (A-30), Equations (A-25) and (A-26) can now be written as follows

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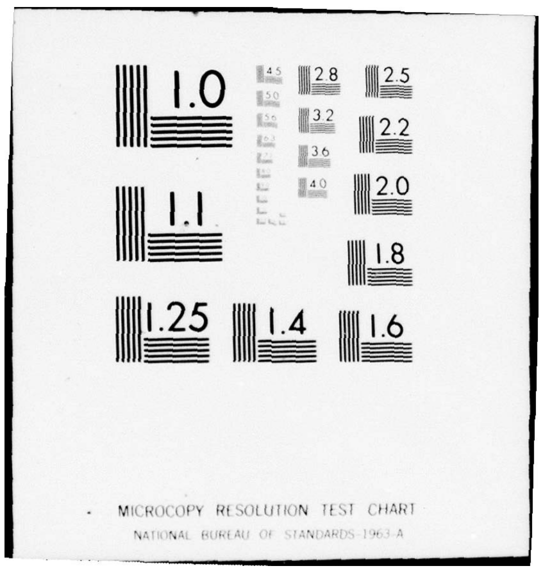
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$$\vec{A} \times \vec{B} = \tilde{A} \cdot \vec{B} = \vec{B} \cdot \tilde{A}^t \qquad \text{(A-31)}$$

$$= \tilde{B}^t \cdot \vec{A} = \vec{A} \cdot \tilde{B}$$

The identity dyadic $\overleftrightarrow{E}$ is defined such that for any vector $\vec{A}$:

$$\vec{A} \cdot \overleftrightarrow{E} = \vec{A} = \overleftrightarrow{E} \cdot \vec{A} \qquad \text{(A-32)}$$

$\overleftrightarrow{E}$ can be expressed in terms of $\hat{x}$, $\hat{y}$, $\hat{z}$ as follows

$$\overleftrightarrow{E} = Q^t Q = \hat{x}\hat{x} + \hat{y}\hat{y} + \hat{z}\hat{z} \qquad \text{(A-33)}$$

If we write $\overleftrightarrow{E} = Q^t E_Q Q$ then $E_Q = 1_3$, the 3×3 unit matrix. The zero dyadic $\overleftrightarrow{O}$ is defined such that for any vector $\vec{A}$:

$$\vec{A} \cdot \overleftrightarrow{O} = \vec{0} = \overleftrightarrow{O} \cdot \vec{A} \qquad \text{(A-34)}$$

If we write $\overleftrightarrow{O} = Q^t O_Q Q$ then O_Q is the 3×3 zero matrix.

We can write the well-known vector triple product identity in the following form:

$$\tilde{A} \cdot \tilde{B} \cdot C = \vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C} \qquad \text{(A-35)}$$

$$= [\vec{B}\vec{A} - (\vec{B} \cdot \vec{A})\overleftrightarrow{E}] \cdot \vec{C}$$

From this it follows that

$$\tilde{A} \cdot \tilde{B} = \vec{B}\vec{A} - (\vec{B} \cdot \vec{A})\overleftrightarrow{E} \qquad \text{(A-36)}$$

Next we write

$$\widetilde{(\vec{A} \times \vec{B})} \cdot \vec{C} = (\vec{A} \times \vec{B}) \times \vec{C}$$
$$= -\vec{C} \times (\vec{A} \times \vec{B})$$
$$= \vec{C} \times (\vec{B} \times \vec{A})$$
$$= (\vec{C} \cdot \vec{A})\vec{B} - (\vec{C} \cdot \vec{B})\vec{A} \qquad \text{(A-37)}$$
$$= \vec{C} \cdot [\vec{A}\vec{B} - \vec{B}\vec{A}]$$
$$= [\vec{B}\vec{A} - \vec{A}\vec{B}] \cdot \vec{C}$$

In the last line we made use of the fact that the transpose of $\vec{A}\vec{B}$ is $\vec{B}\vec{A}$:

$$\overleftrightarrow{D} = \vec{A}\vec{B} \qquad \text{implies } \overleftrightarrow{D}^t = \vec{B}\vec{A} \qquad \text{(A-38)}$$

Hence we have

$$\widetilde{\vec{A} \times \vec{B}} = \vec{B}\vec{A} - \vec{A}\vec{B} \qquad \text{(A-39)}$$
$$= \widetilde{A} \cdot \widetilde{B} - \widetilde{B} \cdot \widetilde{A}$$

Triadics

A *triadic* $\overleftrightarrow{T}$ has the general form

$$\begin{aligned} \overleftrightarrow{T} = \; & T_{xxx}\hat{x}\hat{x}\hat{x} + T_{xxy}\hat{x}\hat{x}\hat{y} + T_{xxz}\hat{x}\hat{x}\hat{z} \\ & + T_{xyx}\hat{x}\hat{y}\hat{x} + T_{xyy}\hat{x}\hat{y}\hat{y} + T_{xyz}\hat{x}\hat{y}\hat{z} \\ & + T_{xzx}\hat{x}\hat{z}\hat{x} + T_{xzy}\hat{x}\hat{z}\hat{y} + T_{xzz}\hat{x}\hat{z}\hat{z} \qquad \text{(A-40)} \\ & + T_{yxx}\hat{y}\hat{x}\hat{x} + \ldots \\ & + \ldots \\ & + T_{zxx}\hat{z}\hat{x}\hat{x} + \ldots \end{aligned}$$

A special triadic is $\overset{\uparrow}{\overleftrightarrow{Z}}$ defined as follows

$$\overset{\uparrow}{\overleftrightarrow{Z}} = -\hat{z}\hat{x}\hat{y} + \hat{y}\hat{x}\hat{z} + \hat{z}\hat{y}\hat{x} \qquad \text{(A-41)}$$
$$-\hat{x}\hat{y}\hat{z} - \hat{y}\hat{z}\hat{x} + \hat{x}\hat{z}\hat{y}$$

By direct expansion we find

$$\vec{A} \cdot \overset{\uparrow}{\overleftrightarrow{Z}} = \tilde{A} \qquad \text{(A-42)}$$

$$\overset{\uparrow}{\overleftrightarrow{Z}} \cdot \vec{B} = \tilde{B} \qquad \text{(A-43)}$$

$$\vec{A} \cdot \overset{\uparrow}{\overleftrightarrow{Z}} \cdot \vec{B} = \vec{A} \times \vec{B} \qquad \text{(A-44)}$$

Thus the triadic $\overset{\uparrow}{\overleftrightarrow{Z}}$ permits us to give equal treatment to $\vec{A}$ and $\vec{B}$ in $\vec{A} \times \vec{B}$; i.e., we can let both of them be vectors, rather than converting one of them to a dyadic.

Let ϵ_{ijk} be the Levi-Civita symbols defined as follows

$$\epsilon_{ijk} = \begin{cases} +1, & \text{if } (i, j, k) \text{ is an even permutation of } (1, 2, 3) \\ -1, & \text{if } (i, j, k) \text{ is an odd permutation of } (1, 2, 3) \\ 0, & \text{otherwise (if two or more of } i, j, \text{ or } k \text{ are equal)} \end{cases} \qquad \text{(A-45)}$$

Also let

$$Q_e = \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{bmatrix} = \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} = Q \qquad \text{(A-46)}$$

$\overleftrightarrow{Z}$ can now be expressed as

$$\overleftrightarrow{Z} = - \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} \epsilon_{ijk} \hat{e}_i \hat{e}_j \hat{e}_k \tag{A-47}$$

Note that $\overleftrightarrow{E}$ is similiarly related to the Kronecker symbol δ_{ij} where

$$\delta_{ij} = \begin{cases} 1, \text{ if } i = j \\ 0, \text{ otherwise (if } i \neq j) \end{cases} \tag{A-48}$$

Evidently, $\overleftrightarrow{E}$ can be written as

$$\overleftrightarrow{E} = \sum_{i=1}^{3} \sum_{j=1}^{3} \delta_{ij} \hat{e}_i \hat{e}_j \tag{A-49}$$

Time Derivatives

We will now obtain expressions for the inertial time derivatives of vectors and dyadics in terms of the time derivatives with respect to a rotating frame. Consider a frame denoted as "frame B" with rectangular, orthogonal, right-handed unit vectors $\hat{x}_B$, $\hat{y}_B$, $\hat{z}_B$. Then the vector $\vec{V}$ can be written as

$$\vec{V} = Q_B^t V_{Q_B} \tag{A-50}$$

where

$$Q_B = \begin{bmatrix} \hat{x}_B \\ \hat{y}_B \\ \hat{z}_B \end{bmatrix} \tag{A-51}$$

and

$$V_{Q_B} = \begin{bmatrix} V_{x_B} \\ V_{y_B} \\ V_{z_B} \end{bmatrix} \tag{A-52}$$

If the inertial angular velocity of frame B is $\vec{\omega}^B$, then

$$\dot{\hat{x}}_B = \vec{\omega}^B \times \hat{x}_B = \tilde{\omega}^B \cdot \hat{x}_B = \hat{x}_B \cdot \tilde{\omega}^{B^t} \tag{A-53}$$

and similiarly for $\dot{\hat{y}}_B$ and $\dot{\hat{z}}_B$. Hence

$$\begin{aligned} \dot{Q}_B^t &= \begin{bmatrix} \dot{\hat{x}}_B & \dot{\hat{y}}_B & \dot{\hat{z}}_B \end{bmatrix} = \tilde{\omega}^B \cdot \begin{bmatrix} \hat{x}_B & \hat{y}_B & \hat{z}_B \end{bmatrix} \\ &= \tilde{\omega}^B \cdot Q_B^t \end{aligned} \tag{A-54}$$

Taking the inertial time derivative of $\vec{V}$ now yields

$$\begin{aligned} \dot{\vec{V}} &= \dot{Q}_B^t V_{Q_B} + Q_B^t \dot{V}_{Q_B} \\ &= \tilde{\omega}^B \cdot Q_B^t V_{Q_B} + Q_B^t \dot{V}_{Q_B} \\ &= \tilde{\omega}^B \cdot \vec{V} + \overset{B}{\dot{\vec{V}}} \end{aligned} \tag{A-55}$$

where $\overset{B}{\vec{V}}$ is the time derivative of $\vec{V}$ with respect to frame B (i.e. the time derivative obtained if frame B is considered fixed). Thus, we have

$$\dot{\vec{V}} = \overset{B}{\vec{V}} + \vec{\omega} \times \vec{V} \tag{A-56}$$

For a dyadic $\overleftrightarrow{D}$

$$\overleftrightarrow{D} = Q_B^t \, D_{Q_B} \, Q_B \tag{A-57}$$

we get similiarly

$$\begin{aligned} \dot{\overleftrightarrow{D}} &= \dot{Q}_B^t \, D_{Q_B} \, Q_B + Q_B^t \, D_{Q_B} \, \dot{Q}_B + Q_B^t \, \dot{D}_{Q_B} \, Q_B \\ &= \tilde{\omega}^B \cdot \overleftrightarrow{D} + \overleftrightarrow{D} \cdot \tilde{\omega}^{B^t} + \overset{B}{\overleftrightarrow{D}} \end{aligned} \tag{A-58}$$

where we have used the transpose of Equation (A-54):

$$\dot{Q}_B = Q_B \cdot \tilde{\omega}^{B^t} \tag{A-59}$$

and we have defined $\overset{B}{\overleftrightarrow{D}}$ as the time derivative of $\overleftrightarrow{D}$ with respect to frame B. Evidently, the time derivative of the identity dyadic is zero

$$\dot{\overleftrightarrow{E}} = \overleftrightarrow{O} \tag{A-60}$$

Note that Equation (A-55) has one term in $\vec{\omega}^B$, and Equation (A-58) has two terms in $\vec{\omega}^B$. Clearly, for the inertial time derivative of a triadic we will get three terms in $\vec{\omega}^B$.

B. MATRICES OF DYADICS, VECTORS, AND SCALARS

It is often convenient to mix matrices of scalars, vectors, and dyadics. For example, Equation (A-22) can be written in matrix form as

$$V = D_Q Q \tag{B-1}$$

where Q and D_Q are as in Equation (A-5) and (A-10), and V is given by

$$V = \begin{bmatrix} \vec{V}_x \\ \vec{V}_y \\ \vec{V}_z \end{bmatrix} \tag{B-2}$$

Now let V^t be defined by

$$V^t = \begin{bmatrix} \vec{V}_x & \vec{V}_y & \vec{V}_z \end{bmatrix} \tag{B-3}$$

Then direct expansion shows that

$$V^t = Q^t D_Q^t \tag{B-4}$$

But this result can also be obtained directly from Equation (B-1) by just formally taking transposes. The validity of such formal manipulations must be justified, but the justification merely involves direct expansion.

Suppose V and W are column matrices of two vectors and two scalars

$$V = \begin{bmatrix} \vec{V}_1 \\ \vec{V}_2 \\ V_3 \\ V_4 \end{bmatrix} \quad ; \qquad W = \begin{bmatrix} \vec{W}_1 \\ \vec{W}_2 \\ W_3 \\ W_4 \end{bmatrix} \tag{B-5}$$

Here $\vec{V}_1$, $\vec{V}_2$, $\vec{W}_1$, $\vec{W}_2$ are vectors and V_3, V_4, W_3, W_4 are scalars. Next suppose we have four equations as follows

$$\begin{aligned}
\vec{V}_1 &= \overleftrightarrow{A}_{11} \cdot \vec{W}_1 + \overleftrightarrow{A}_{12} \cdot \vec{W}_2 + \vec{A}_{13} W_3 + \vec{A}_{14} W_4 \\
\vec{V}_2 &= \overleftrightarrow{A}_{21} \cdot \vec{W}_1 + \overleftrightarrow{A}_{22} \cdot \vec{W}_2 + \vec{A}_{23} W_3 + \vec{A}_{24} W_4 \\
V_3 &= \vec{A}_{31} \cdot \vec{W}_1 + \vec{A}_{32} \cdot \vec{W}_2 + A_{33} W_3 + A_{34} W_4 \\
V_4 &= \vec{A}_{41} \cdot \vec{W}_1 + \vec{A}_{42} \cdot \vec{W}_2 + A_{43} W_3 + A_{44} W_4
\end{aligned} \tag{B-6}$$

It is evident that we can put these equations in matrix form as follows:

$$\begin{bmatrix} \vec{V}_1 \\ \vec{V}_2 \\ V_3 \\ V_4 \end{bmatrix} = \begin{bmatrix} \overleftrightarrow{A}_{11} & \overleftrightarrow{A}_{12} & \vec{A}_{13} & \vec{A}_{14} \\ \overleftrightarrow{A}_{21} & \overleftrightarrow{A}_{22} & \vec{A}_{23} & \vec{A}_{24} \\ \vec{A}_{31} & \vec{A}_{32} & A_{33} & A_{34} \\ \vec{A}_{41} & \vec{A}_{42} & A_{43} & A_{44} \end{bmatrix} \bullet \begin{bmatrix} \vec{W}_1 \\ \vec{W}_2 \\ W_3 \\ W_4 \end{bmatrix} \tag{B-7}$$

Since $\overleftrightarrow{A}_{11} \cdot \vec{W}_1 = \vec{W}_1 \cdot \overleftrightarrow{A}^t_{11}$, $\overleftrightarrow{A}_{12} \cdot \vec{W}_2 = \vec{W}_2 \cdot \overleftrightarrow{A}^t_{12}$, etc., we can

also write this as

$$\left[\vec{V}_1 \quad \vec{V}_2 \quad V_3 \quad V_4\right] =$$

$$= \left[\vec{W}_1 \quad \vec{W}_2 \quad W_3 \quad W_4\right] \cdot \begin{bmatrix} \overleftrightarrow{A}_{11}^{t} & \overleftrightarrow{A}_{21}^{t} & \vec{A}_{31} & \vec{A}_{41} \\ \overleftrightarrow{A}_{12}^{t} & \overleftrightarrow{A}_{22}^{t} & \vec{A}_{32} & \vec{A}_{42} \\ \vec{A}_{13} & \vec{A}_{23} & A_{33} & A_{43} \\ \vec{A}_{14} & \vec{A}_{24} & A_{34} & A_{44} \end{bmatrix} \qquad \text{(B-8)}$$

Thus Equations (B-7) and (B-8) can be written as

$$V = A \cdot W \qquad \text{(B-9)}$$

and

$$V^t = W^t \cdot A^t \qquad \text{(B-10)}$$

where A and A^t are the matrices in Equations (B-7) and (B-8), respectively. Note that the dyadic elements of A were changed to their transposes when A was changed to A^t.

In order to have a uniform procedure for forming transposes of matrices whose elements may be dyadics, vectors, or scalars, we adopt the following convention: the transpose of a scalar and of a vector are equal to the scalar, and vector, respectively:

$$\begin{aligned} s^t &= s && \text{(scalar s)} \\ \vec{V}^t &= \vec{V} && \text{(vector } \vec{V}\text{)} \end{aligned} \qquad \text{(B-11)}$$

Consequently, A^t is formed from A by taking the element in the i^{th} row and j^{th} column of A and putting the transpose of this element in the j^{th} row and i^{th} column of A^t. Note that the same procedure applies to forming V^t and W^t from V and W, respectively.

Suppose X is a column matrix of n vectors

$$X = \begin{bmatrix} \vec{X}_1 \\ \vec{X}_2 \\ \cdot \\ \cdot \\ \cdot \\ \vec{X}_n \end{bmatrix} \tag{B-12}$$

We now define $\widetilde{X}$ by

$$\widetilde{X} = \begin{bmatrix} \widetilde{X}_1 \\ \widetilde{X}_2 \\ \cdot \\ \cdot \\ \cdot \\ \widetilde{X}_n \end{bmatrix} \tag{B-13}$$

Note that $\widetilde{X}_j$ was defined in Appendix A for any vector $\vec{X}_j$. Now we have defined $\widetilde{X}$ for any column matrix of vectors. Next, we note that our above convention on transposes requires that the transpose of $\widetilde{X}$ must be

$$\tilde{X}^t = \begin{bmatrix} \tilde{X}_1^t & \tilde{X}_2^t & \cdots & \tilde{X}_n^t \end{bmatrix} \qquad \text{(B-14)}$$

$$= - \begin{bmatrix} \tilde{X}_1 & \tilde{X}_2 & \cdots & \tilde{X}_n \end{bmatrix}$$

Thus, $\tilde{X}^t$ is the negative of what one obtains by first taking the row matrix of vectors X^t and then replacing each of the row elements $\vec{X}_i$ by $\tilde{X}_i$. The potential confusion here due to this change in sign can be eliminating by simply remembering that we start with a column matrix of vectors, then perform the "tilde operation", and only then do we take the transpose. In other words, the "tilde operation" is only defined for a single vector or for a column matrix of vectors (but not for a row matrix of vectors).

From Equation (A-31) we have $\vec{A} \times \vec{B} = \tilde{A} \cdot \vec{B} = \tilde{B}^t \cdot \vec{A}$. A generalization of this is the following

$$\vec{A} \times X = \tilde{A} \cdot X = \tilde{A} \cdot \begin{bmatrix} \vec{X}_1 \\ \vec{X}_2 \\ \cdot \\ \cdot \\ \cdot \\ \vec{X}_n \end{bmatrix} = \vec{A} \cdot \tilde{X} \qquad \text{(B-15)}$$

Also

$$\vec{A} \times X^t = \tilde{A} \cdot X^t = \tilde{A} \cdot \begin{bmatrix} \vec{X}_1 & \vec{X}_2 & \cdots & \vec{X}_n \end{bmatrix}$$
$$= \vec{A} \cdot \begin{bmatrix} \tilde{X}_1 & \tilde{X}_2 & \cdots & \tilde{X}_n \end{bmatrix}$$
$$= \begin{bmatrix} \tilde{X}_1^t & \tilde{X}_2^t & \cdots & \tilde{X}_n^t \end{bmatrix} \cdot \vec{A} \qquad \text{(B-16)}$$
$$= \tilde{X}^t \cdot \vec{A}$$
$$= -\vec{A} \cdot \tilde{X}^t$$

A further generalization is

$$Y \times X^t = \tilde{Y} \cdot X^t = \begin{bmatrix} \tilde{Y}_1 \\ \tilde{Y}_2 \\ \cdot \\ \cdot \\ \cdot \\ \tilde{Y}_n \end{bmatrix} \cdot \begin{bmatrix} \vec{X}_1 & \vec{X}_2 & \cdots & \vec{X}_n \end{bmatrix}$$
$$= \begin{bmatrix} \vec{Y}_1 \\ \vec{Y}_2 \\ \cdot \\ \cdot \\ \cdot \\ \vec{Y}_n \end{bmatrix} \cdot \begin{bmatrix} \tilde{X}_1 & \tilde{X}_2 & \cdots & \tilde{X}_n \end{bmatrix} \qquad \text{(B-17)}$$
$$= -Y \cdot \tilde{X}^t$$

A somewhat similiar situation occurs if we have a vector $\vec{V}$ given by

$$\vec{V} = \sum_i \vec{X}_i s_i = s^t X = X^t s \qquad \text{(B-18)}$$

where the s_i are scalars and s and s^t are column and row matrices of these scalars. We now form $\tilde{V}$ as follows

$$\tilde{V} = \sum_i \tilde{X}_i s_i = s^t \tilde{X}$$

$$= \begin{bmatrix} s_1 & s_2 & \cdots & s_n \end{bmatrix} \begin{bmatrix} \tilde{X}_1 \\ \tilde{X}_2 \\ \cdot \\ \cdot \\ \cdot \\ \tilde{X}_n \end{bmatrix}$$

$$= \begin{bmatrix} \tilde{X}_1 & \tilde{X}_2 & \cdots & \tilde{X}_n \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ \cdot \\ \cdot \\ \cdot \\ s_n \end{bmatrix} \qquad \text{(B-19)}$$

$$= -\tilde{X}^t \mathbf{s}$$

Equations (B-16), (B-17), and (B-19) have negative signs because of the fact that the "tilde operation" and the transpose operation do not commute (in fact, they anti-commute).

Time derivatives of column matrices of vectors and row

matrices of vectors are easily obtained from Equation (A-55) or (A-56). We find

$$\dot{X} = \overset{B}{X} + \tilde{\omega}^{B} \cdot X \qquad \text{(B-20)}$$

$$= \overset{B}{X} + X \cdot \tilde{\omega}^{B^t}$$

and

$$\dot{X}^t = \overset{B}{X}{}^t + \tilde{\omega}^{B} \cdot X^t \qquad \text{(B-21)}$$

$$= \overset{B}{X}{}^t + X^t \cdot \tilde{\omega}^{B^t}$$

where $\vec{\omega}^B$ is the inertial angular velocity of a particular reference frame (called frame B), and $\overset{B}{X}$ is the column matrix of vectors whose j^{th} element is $\overset{B}{\vec{X}}_j$ (if X is as given in Equation (B-12)), which is the time derivative of $\vec{X}_j$ with respect to this particular reference frame.

C. DETAILED EVALUATION OF $\bar{D}$ AND $\dot{\bar{\mu}}$

We will now apply the formalism of the previous two appendices and derive the results given in Section IV D. Before doing this we recall the following definition of $\bar{D}$ given in Section IV C.

$$\bar{D} = \begin{bmatrix} \overleftrightarrow{O} & \overleftrightarrow{O} & 0 \\ M\dot{\tilde{R}}_{ca} & \bar{D}_{22} & \bar{D}_{23} \\ \dot{\alpha}^{B}_{a} & \bar{D}_{32} & \bar{D}_{33} \end{bmatrix} \tag{C-1}$$

where

$$M\dot{\tilde{R}}_{ca} = \sum_i m_i \dot{\tilde{R}}_{ia} \tag{C-2}$$

$$\dot{\alpha}^{B}_{a} = \sum_i m_i \dot{\Phi}^{B}_{ia} \tag{C-3}$$

$$\bar{D}_{22} = \sum_i m_i \dot{\tilde{R}}_{ia} \cdot \tilde{R}^{t}_{ia} \tag{C-4}$$

$$\bar{D}_{32} = \sum_i m_i \dot{\Phi}^{B}_{ia} \cdot \tilde{R}^{t}_{ia} \tag{C-5}$$

$$\bar{D}_{23} = \sum_i m_i \dot{\tilde{R}}_{ia} \cdot \Phi^{B^t}_{ia} \tag{C-6}$$

$$\bar{D}_{33} = \sum_i m_i \dot{\Phi}^{B}_{ia} \cdot \Phi^{B^t}_{ia} \tag{C-7}$$

In addition there are the relationships

$$\dot{\overrightarrow{I}}_a = \bar{D}_{22} + \bar{D}_{22}^{\,t} = \dot{\overleftrightarrow{I}}^{t}_{a} \tag{C-8}$$

$$\dot{\beta}_a^B = \bar{D}_{32} + \bar{D}_{23}^t \tag{C-9}$$

$$\dot{\gamma}_a^B = \bar{D}_{33} + \bar{D}_{33}^t = \dot{\gamma}_a^{B^t} \tag{C-10}$$

where

$$\overleftrightarrow{I}_a = \sum_i m_i \tilde{R}_{ia} \cdot \tilde{R}_{ia}^t = \overleftrightarrow{I}_a^t \tag{C-11}$$

$$\beta_a^B = \sum_i m_i \Phi_{ia}^B \cdot \tilde{R}_{ia}^t = \sum_i m_i \tilde{R}_{ia} \cdot \Phi_{ia}^B \tag{C-12}$$

$$\gamma_a^B = \sum_i m_i \Phi_{ia}^B \cdot \Phi_{ia}^{B^t} = \gamma_a^{B^t} \tag{C-13}$$

In these equations the summation is over all particles (from i = 1 to N) and the i^{th} particle is at position $\vec{R}_{ia}$ relative to the arbitrary reference point a. $\dot{\vec{R}}_{ia}$ is the inertial time derivative of $\vec{R}_{ia}$ and can be expressed as follows

$$\dot{\vec{R}}_{ia} = \overset{B}{\vec{R}}_{ia} + \vec{\omega}^B \times \vec{R}_{ia} \tag{C-14}$$

$$= \dot{\xi}_a^{B^t} \Phi_{ia}^B + \tilde{\omega}^B \cdot \vec{R}_{ia}$$

where $\overset{B}{\vec{R}}_{ia}$ is the time derivative of $\vec{R}_{ia}$ with respect to frame B which has angular velocity $\vec{\omega}^B$ with respect to inertial space. $\dot{\xi}_a^{B^t}$ is a row matrix of scalars, each of these scalars being the time derivative of an internal generalized coordinate ξ_{aj}^B:

$$\dot{\xi}_a^{B^t} = \left[\dot{\xi}_{a1}^B \quad \dot{\xi}_{a2}^B \quad \cdots \quad \dot{\xi}_{an}^B\right] \tag{C-15}$$

Φ^B_{ia} is a column matrix of vectors

$$\Phi^B_{ia} = \begin{bmatrix} \vec{\Phi}^B_{ia1} \\ \vec{\Phi}^B_{ia2} \\ \cdot \\ \cdot \\ \cdot \\ \vec{\Phi}^B_{ian} \end{bmatrix} \qquad \text{(C-16)}$$

where

$$\vec{\Phi}^B_{iaj} = \frac{\partial \vec{R}_{ia}}{\partial \xi^B_{aj}} \qquad \text{(C-17)}$$

Thus, Φ^B_{ia} is a column matrix of *the partial derivatives of* $\vec{R}_{ia}$ *with* respect to the internal generalized coordinates. This fact can be brought out more clearly by introducing the operator $\nabla_{\xi^B_a}$ as follows:

$$\nabla_{\xi^B_a} = \begin{bmatrix} \dfrac{\partial}{\partial \xi^B_{a1}} \\ \dfrac{\partial}{\partial \xi^B_{a2}} \\ \cdot \\ \cdot \\ \cdot \\ \dfrac{\partial}{\partial \xi^B_{an}} \end{bmatrix} \qquad \text{(C-18)}$$

Equations (C-16) to (C-18) can now be combined as follows:

$$\Phi^B_{ia} = \nabla_{\xi^B_a} \vec{R}_{ia} \tag{C-19}$$

We define the operation of $\nabla_{\xi^B_a}$ on a row matrix of vectors as follows

$$\begin{aligned}\nabla_{\xi^B_a} \Phi^{B^t}_{ia} &= \nabla_{\xi^B_a} \left[\vec{\Phi}^B_{ia1} \quad \vec{\Phi}^B_{ia2} \quad \cdots \quad \vec{\Phi}^B_{ian}\right] \\ &= \left[\nabla_{\xi^B_a} \vec{\Phi}^B_{ia1} \quad \nabla_{\xi^B_a} \vec{\Phi}^B_{ia2} \quad \cdots \quad \nabla_{\xi^B_a} \vec{\Phi}^B_{ian}\right]\end{aligned} \tag{C-20}$$

Note that $\nabla_{\xi^B_a} \Phi^{B^t}_{ia}$ is an n x n matrix of vectors and the element in the j^{th} row and k^{th} column is

$$\left(\nabla_{\xi^B_a} \Phi^{B^t}_{ia}\right)_{jk} = \frac{\partial \vec{\Phi}_{iak}}{\partial \xi^B_{aj}} = \frac{\partial^2 \vec{R}_{ia}}{\partial \xi^B_{aj} \, \partial \xi^B_{ak}} \tag{C-21}$$

The transpose of $\nabla_{\xi^B_a} \Phi^{B^t}_{ia}$ is

$$\left(\nabla_{\xi^B_a} \Phi^{B^t}_{ia}\right)^t = \begin{bmatrix} \left(\nabla_{\xi^B_a} \vec{\Phi}^B_{ia1}\right)^t \\ \left(\nabla_{\xi^B_a} \vec{\Phi}^B_{ia2}\right)^t \\ \vdots \\ \left(\nabla_{\xi^B_a} \vec{\Phi}^B_{ian}\right)^t \end{bmatrix} \tag{C-22}$$

and the element in the j^{th} row and k^{th} column is

$$\left(\left(\nabla_{\xi^B_a} \Phi^{B^t}_{ia}\right)^t\right)_{jk} = \frac{\partial \vec{\Phi}^B_{iaj}}{\partial \xi^B_{ak}} = \frac{\partial^2 \vec{R}_{ia}}{\partial \xi^B_{ak}\,\partial \xi^B_{aj}} \tag{C-23}$$

Comparing Equations (C-21) and (C-23) we note that if we can interchange the order of the partial derivatives then the matrix $\nabla_{\xi^B_a} \Phi^{B^t}_{ia}$ is symmetric. The interchange of the order of the partial derivatives is permissible if $\vec{R}_{ia}$ and its first two partial derivatives are continuous (i.e. are continuous functions of the internal generalized coordinates).

Since Φ^B_{ia} is a column matrix of vectors we have

$$\dot{\Phi}^B_{ia} = \overset{B}{\dot{\Phi}}{}^B_{ia} + \tilde{\omega}^B \cdot \Phi^B_{ia} \tag{C-24}$$

where ${}^{B}\dot{\Phi}_{ia}^{B}$ is the time derivative of Φ_{ia}^{B} in frame B. But we can write

$$ {}^{B}\dot{\Phi}_{ia}^{B} = \sum_{k=1}^{n} \frac{\partial \Phi_{ia}^{B}}{\partial \xi_{ak}^{B}} \dot{\xi}_{ak}^{B} \qquad \text{(C-25)} $$

This expression can be written in terms of $\nabla_{\xi_a^B}$ if we first take the transpose

$$ {}^{B}\dot{\Phi}_{ia}^{B^t} = \sum_{k=1}^{n} \dot{\xi}_{ak}^{B} \frac{\partial \Phi_{ia}^{B^t}}{\partial \xi_{ak}^{B}} = \dot{\xi}_{a}^{B^t} \nabla_{\xi_a^B} \Phi_{ia}^{B^t} \qquad \text{(C-26)} $$

Taking the transpose of this and combining with Equation (C-24) yields

$$ \dot{\Phi}_{ia}^{B} = \left(\nabla_{\xi_a^B} \Phi_{ia}^{B^t} \right)^t \dot{\xi}_{a}^{B} + \tilde{\omega}^{B} \cdot \Phi_{ia}^{B} \qquad \text{(C-27)} $$

We will use this equation in Equations (C-2) to (C-13), as required.

In order to evaluate Equations (C-2) to (C-10) we also need an expression for $\dot{\tilde{R}}_{ia}$. Now from Equations (B-18) and (B-19) we have

$$ {}^{B}\dot{\tilde{R}}_{ia} = \dot{\xi}_{a}^{B^t} \tilde{\Phi}_{ia}^{B} \qquad \text{(C-28)} $$

where

$$\tilde{\Phi}_{ia}^{B} = \begin{bmatrix} \tilde{\Phi}_{ia1}^{B} \\ \tilde{\Phi}_{ia2}^{B} \\ \cdot \\ \cdot \\ \cdot \\ \tilde{\Phi}_{ian}^{B} \end{bmatrix} \tag{C-29}$$

From Equation (A-39) we have

$$\widetilde{\vec{\omega}^{B} \times \vec{R}_{ia}} = \tilde{\omega}^{B} \cdot \tilde{R}_{ia} - \tilde{R}_{ia} \cdot \tilde{\omega}^{B} \tag{C-30}$$

Combining Equations (C-14), (C-28), and (C-30) now yields

$$\dot{\tilde{R}}_{ia} = \dot{\xi}_{a}^{B^t} \tilde{\Phi}_{ia}^{B} + \tilde{\omega}^{B} \cdot \tilde{R}_{ia} - \tilde{R}_{ia} \cdot \tilde{\omega}^{B} \tag{C-31}$$

We are now ready to evaluate Equations (C-2) to (C-10).

<u>Evaluation of $M\dot{\tilde{R}}_{ca}$</u>

Multiplying $\dot{\tilde{R}}_{ia}$ by m_i and summing over all the particles yields immediately

$$M\dot{\tilde{R}}_{ca} = \sum_{i} m_i \dot{\tilde{R}}_{ia} = \dot{\xi}_{a}^{B^t} \tilde{\alpha}_{a}^{B} + \tilde{\omega}^{B} \cdot M\tilde{R}_{ca} - M\tilde{R}_{ca} \cdot \tilde{\omega}^{B} \tag{C-32}$$

where we have made use of the expression

$$\alpha_{a}^{B} = \sum_{i} m_i \Phi_{ia}^{B} \tag{C-33}$$

from which it follows that

$$\tilde{\alpha}_a^B = \sum_i m_i \tilde{\Phi}_{ia}^B \tag{C-34}$$

Evaluation of $\bar{D}_{22}$

From Equation (C-4) and (C-31):

$$\bar{D}_{22} = \sum_i m_i \dot{\tilde{R}}_{ia} \cdot \tilde{R}_{ia}^t = \dot{\xi}_a^{B^t} \sum_i m_i \tilde{\Phi}_{ia}^B \cdot \tilde{R}_{ia}^t + \tilde{\omega}^B \cdot \overleftrightarrow{I}_a - \sum_i m_i \tilde{R}_{ia} \cdot \tilde{\omega}^B \cdot \tilde{R}_{ia}^t \tag{C-35}$$

where we have introduced $\overleftrightarrow{I}_a$ from Equation (C-11). The last term on the right hand side of Equation (C-35) is not in an acceptable form since we want to separate the terms involving the individual particles (i.e. terms involving the particle subscripts) from terms like $\vec{\omega}^B$ which do not involve the individual particles. In order to do this we expand as follows, making use of Equation (A-36).

$$\begin{aligned} \tilde{R}_{ia} \cdot \tilde{\omega}^B \cdot \tilde{R}_{ia}^t &= \tilde{R}_{ia} \cdot \left[\left(\vec{R}_{ia} \cdot \vec{\omega}^B\right)\overleftrightarrow{E} - \vec{R}_{ia}\vec{\omega}^B\right] \\ &= \left(\vec{R}_{ia} \cdot \vec{\omega}^B\right)\tilde{R}_{ia} - \tilde{R}_{ia} \cdot \vec{R}_{ia}\vec{\omega}^B \end{aligned} \tag{C-36}$$

But the last dyadic on the right is zero because if we dot multiply it into an arbitrary vector $\vec{V}$, we get $\tilde{R}_{ia} \cdot \vec{R}_{ia}\left(\vec{\omega}^B \cdot \vec{V}\right) = \vec{0}$ because $\vec{R}_{ia} \times \vec{R}_{ia} = \vec{0}$. Consequently we have

$$\tilde{R}_{ia} \cdot \tilde{\omega}^B \cdot \tilde{R}_{ia}^t = \vec{\omega}^B \cdot \vec{R}_{ia}\tilde{R}_{ia} = \tilde{R}_{ia}\vec{R}_{ia} \cdot \vec{\omega}^B \tag{C-37}$$

Note that $\vec{R}_{ia}\widetilde{R}_{ia}$ and $\widetilde{R}_{ia}\vec{R}_{ia}$ are triadics. $\overline{D}_{22}$ can now be expressed as

$$\overline{D}_{22} = \sum_i m_i \dot{\widetilde{R}}_{ia} \cdot \widetilde{R}^t_{ia} = \dot{\xi}^{B^t}_a \sum_i m_i \widetilde{\Phi}^B_{ia} \cdot \widetilde{R}^t_{ia} + \widetilde{\omega}^B \cdot \overleftrightarrow{I}_a - \vec{\omega}^B \cdot \sum_i m_i \vec{R}_{ia}\widetilde{R}_{ia}$$

(C-38)

<u>Evaluation of $\overline{D}_{23}$</u>

From Equation (C-6) and (C-31):

$$\overline{D}_{23} = \sum_i m_i \dot{\widetilde{R}}_{ia} \cdot \Phi^{B^t}_{ia} = \dot{\xi}^{B^t}_a \sum_i m_i \widetilde{\Phi}^B_{ia} \cdot \Phi^{B^t}_{ia} + \widetilde{\omega}^B \cdot \beta^{B^t}_a - \sum_i m_i \widetilde{R}_{ia} \cdot \widetilde{\omega}^B \cdot \Phi^{B^t}_{ia}$$

(C-39)

where we have introduced β^B_a from Equation (C-12). Again, the last term on the right must be manipulated so that $\vec{\omega}^B$ can be factored out. In order to this we use the following from Equation (B-16)

$$\widetilde{\omega}^B \cdot \Phi^{B^t}_{ia} = \widetilde{\Phi}^{B^t}_{ia} \cdot \vec{\omega}^B \qquad \text{(C-40)}$$

Analogously

$$\widetilde{\omega}^B \cdot \beta^{B^t}_a = \widetilde{\beta}^{B^t}_a \cdot \vec{\omega}^B \qquad \text{(C-41)}$$

But we can write

$$\widetilde{\beta}^{B^t}_a = \sum_i m_i \widetilde{R}_{ia} \cdot \widetilde{\Phi}^{B^t}_{ia} - \sum_i m_i \widetilde{\Phi}^{B^t}_{ia} \cdot \widetilde{R}_{ia} \qquad \text{(C-42)}$$

From Equations (C-40) to (C-42) we now have

$$\tilde{\omega}^B \cdot \beta_a^{B^t} - \sum_i m_i \tilde{R}_{ia} \cdot \tilde{\omega}^B \cdot \Phi_{ia}^{B^t} = \left[\tilde{\beta}_a^{B^t} - \sum_i m_i \tilde{R}_{ia} \cdot \tilde{\Phi}_{ia}^{B^t}\right] \cdot \vec{\omega}^B$$
$$= -\sum_i m_i \tilde{\Phi}_{ia}^{B^t} \cdot \tilde{R}_{ia} \cdot \vec{\omega}^B \tag{C-43}$$

$\bar{D}_{23}$ can now be expressed as

$$\bar{D}_{23} = \sum_i m_i \dot{\tilde{R}}_{ia} \cdot \Phi_{ia}^{B^t} = \dot{\xi}_a^{B^t} \sum_i m_i \tilde{\Phi}_{ia}^B \cdot \Phi_{ia}^{B^t} + \sum_i m_i \tilde{\Phi}_{ia}^{B^t} \cdot \tilde{R}_{ia}^t \cdot \vec{\omega}^B \tag{C-44}$$

<u>Explicit Verification of Equation (4-81)</u>

We can use Equation (4-81) of Section IV C to check the correctness of the above explicit expressions for $\bar{D}_{22}$ and $\bar{D}_{23}$. When we dot multiply $\bar{D}_{22}$ of Equation (C-38) into $\vec{\omega}^B$ we get three terms, and the second and third terms cancel, as can be seen from the following

$$\tilde{\omega}^B \cdot \overleftrightarrow{I}_a \cdot \vec{\omega}^B = \tilde{\omega}^B \cdot \sum_i m_i \tilde{R}_{ia} \cdot \tilde{R}_{ia}^t \cdot \vec{\omega}^B$$
$$= \tilde{\omega}^B \cdot \sum_i m_i \left[\left(\vec{R}_{ia} \cdot \vec{R}_{ia}\right)\overleftrightarrow{E} - \vec{R}_{ia}\vec{R}_{ia}\right] \cdot \vec{\omega}^B$$
$$= -\tilde{\omega}^B \cdot \sum_i m_i \vec{R}_{ia}\vec{R}_{ia} \cdot \vec{\omega}^B$$
$$= \sum_i m_i \left(\vec{R}_{ia}\vec{R}_{ia} \cdot \vec{\omega}^B\right) \cdot \tilde{\omega}^B$$
$$= \vec{\omega}^B \cdot \sum_i m_i \vec{R}_{ia}\vec{R}_{ia} \cdot \tilde{\omega}^B$$
$$= \vec{\omega}^B \cdot \sum_i m_i \vec{R}_{ia}\tilde{R}_{ia} \cdot \vec{\omega}^B \tag{C-45}$$

Hence, $\bar{D}_{22} \cdot \vec{\omega}^B$ reduces to

$$\bar{D}_{22} \cdot \vec{\omega}^B = \dot{\xi}^{B^t}_a \sum_i m_i \tilde{\Phi}^B_{ia} \cdot \tilde{R}^t_{ia} \cdot \vec{\omega}^B \qquad \text{(C-46)}$$

When we multiply $\bar{D}_{23}$ of Equation (C-44) into $\dot{\xi}^B_a$ we get two terms but the first term is zero because $\tilde{\Phi}^B_{ia} \cdot \Phi^{B^t}_{ia}$ is an n × n skew-symmetric matrix of vectors, as can be seen from making use of Equation (B-17) which yields

$$\tilde{\Phi}^B_{ia} \cdot \Phi^{B^t}_{ia} = -\Phi^B_{ia} \cdot \tilde{\Phi}^{B^t}_{ia} \qquad \text{(C-47)}$$

Evidently the matrix on the left is equal to the negative of its transpose. From this it follows that

$$\dot{\xi}^{B^t}_a \sum_i m_i \tilde{\Phi}^B_{ia} \cdot \Phi^{B^t}_{ia} \dot{\xi}^B_a = \vec{0} \qquad \text{(C-48)}$$

Hence $\bar{D}_{23} \dot{\xi}^B_a$ reduces to

$$\bar{D}_{23} \dot{\xi}^B_a = \sum_i m_i \tilde{\Phi}^{B^t}_{ia} \cdot \tilde{R}^t_{ia} \cdot \vec{\omega}^B \dot{\xi}^B_a = -\dot{\xi}^{B^t}_a \sum_i m_i \tilde{\Phi}^B_{ia} \cdot \tilde{R}^t_{ia} \cdot \vec{\omega}^B \qquad \text{(C-49)}$$

Here we have made use of Equation (B-19) to yield

$$\tilde{\Phi}^{B^t}_{ia} \dot{\xi}^B_a = -\dot{\xi}^{B^t}_a \tilde{\Phi}^B_{ia} \qquad \text{(C-50)}$$

Equations (C-46) and (C-49) now show explicitly that Equation(4-81) is satisfied; i.e.,

$$\bar{D}_{22} \cdot \vec{\omega}^B + \bar{D}_{23} \dot{\xi}^B_a = \vec{0} \qquad \text{(C-51)}$$

Evaluation of $\dot{\alpha}_a^B$

From Equation (C-3) and (C-27) we immediately get

$$\dot{\alpha}_a^B = \sum_i m_i \dot{\Phi}_{ia}^B = \left(\nabla_{\xi_a^B} \alpha_a^{B^t}\right)^t \dot{\xi}_a^B + \tilde{\omega}^B \cdot \alpha_a^B \qquad \text{(C-52)}$$

where we have made use of Equation (C-33) in the form

$$\nabla_{\xi_a^B} \alpha_a^{B^t} = \sum_i m_i \nabla_{\xi_a^B} \Phi_{ia}^{B^t} \qquad \text{(C-53)}$$

If $\nabla_{\xi_a^B} \Phi_{ia}^{B^t}$ is symmetric (i.e. if $\vec{R}_{ia}$ and its first two partial derivatives are continuous) then $\nabla_{\xi_a^B} \alpha_a^{B^t}$ is also symmetric.

Evaluation of $\bar{D}_{32}$

From Equation (C-5) and (C-27) we get

$$\bar{D}_{32} = \sum_i m_i \dot{\Phi}_{ia}^B \cdot \tilde{R}_{ia}^t = \sum_i m_i \left[\left(\left(\nabla_{\xi_a^B} \Phi_{ia}^{B^t}\right)^t \dot{\xi}_a^B\right)\right] \cdot \tilde{R}_{ia}^t + \sum_i m_i \tilde{\omega}^B \cdot \Phi_{ia}^B \cdot \tilde{R}_{ia}^t$$

(C-54)

In order to separate $\vec{\omega}^B$ from the particle dependent factors in the second term on the right, we use Equation (B-15) in the form

$$\tilde{\omega}^B \cdot \Phi_{ia}^B = \vec{\omega}^B \cdot \tilde{\Phi}_{ia}^B \qquad \text{(C-55)}$$

Consequently, $\bar{D}_{32}$ can be written as

$$\bar{D}_{32} = \sum_i m_i \dot{\Phi}_{ia}^B \cdot \tilde{R}_{ia}^t = \sum_i m_i \tilde{R}_{ia} \cdot \left(\nabla_{\xi_a^B} \Phi_{ia}^{B^t}\right)^t \dot{\xi}_a^B + \vec{\omega}^B \cdot \sum_i m_i \tilde{\Phi}_{ia}^B \cdot \tilde{R}_{ia}^t$$

(C-56)

Evaluation of $\overline{D}_{33}$

For $\overline{D}_{33}$ we obtain from Equations (C-7) and (C-27)

$$\overline{D}_{33} = \sum_i m_i \dot{\Phi}^B_{ia} \cdot \Phi^{B^t}_{ia} = \sum_i m_i \left[\left(\nabla_{\xi^B_a} \Phi^{B^t}_{ia}\right)^t \dot{\xi}^B_a\right] \cdot \Phi^{B^t}_{ia} + \sum_i m_i \widetilde{\omega}^B \cdot \Phi^B_{ia} \cdot \Phi^{B^t}_{ia}$$

$$= \sum_i m_i \left[\left(\nabla_{\xi^B_a} \Phi^{B^t}_{ia}\right)^t \dot{\xi}^B_a\right] \cdot \Phi^{B^t}_{ia} + \vec{\omega}^B \cdot \sum_i m_i \widetilde{\Phi}^B_{ia} \cdot \Phi^{B^t}_{ia} \tag{C-57}$$

where we have again used Equation (C-55) to separate $\vec{\omega}^B$ from the particle dependent factor.

j^{th} Row of $\overline{D}_{32}$ and $\overline{D}_{33}$

Note that $\left(\nabla_{\xi^B_a} \Phi^{B^t}_{ia}\right)^t \dot{\xi}^B_a$ is a column matrix of vectors, and $\overline{D}_{32}$ is also a column matrix of vectors, whereas $\overline{D}_{33}$ is an $n \times n$ matrix of scalars. It is instructive to examine the j^{th} vector of $\overline{D}_{32}$ and the j^{th} row of $\overline{D}_{33}$. In order to do this we note from Equation (C-22) that the j^{th} row of $\left(\nabla_{\xi^B_a} \Phi^{B^t}_{ia}\right)^t$ is $\left(\nabla_{\xi^B_a} \vec{\Phi}^B_{iaj}\right)^t$.

Thus the j^{th} vector of $\overline{D}_{32}$ is

$$(\overline{D}_{32})_j = \sum_i m_i \widetilde{R}_{ia} \cdot \left(\nabla_{\xi^B_a} \vec{\Phi}^B_{iaj}\right)^t \dot{\xi}^B_a + \vec{\omega}^B \cdot \sum_i m_i \widetilde{\Phi}^B_{iaj} \cdot \widetilde{R}^t_{ia} \tag{C-58}$$

and the j^{th} row of $\overline{D}_{33}$ is

$$(\overline{D}_{33})_{j.} = \sum_i m_i \left[\left(\nabla_{\xi^B_a} \vec{\Phi}^B_{iaj}\right)^t \dot{\xi}^B_a\right] \cdot \Phi^{B^t}_{ia} + \vec{\omega}^B \cdot \sum_i m_i \widetilde{\Phi}^B_{iaj} \cdot \Phi^{B^t}_{ia}$$

$$= \dot{\xi}^{B^t}_a \sum_i m_i \left(\nabla_{\xi^B_a} \vec{\Phi}^B_{iaj}\right) \cdot \Phi^{B^t}_{ia} + \vec{\omega}^B \cdot \sum_i m_i \widetilde{\Phi}^B_{iaj} \cdot \Phi^{B^t}_{ia} \tag{C-59}$$

Note that by writing out only the j^{th} row of $\bar{D}_{33}$ (rather than the whole matrix) it is possible to separate $\dot{\xi}_a^{B^t}$ from the particle dependent factor.

Evaluation of $\overleftrightarrow{I}_a$

According to Equation (C-8) $\overleftrightarrow{I}_a$ is equal to the sum of $\bar{D}_{22}$ and its transpose. $\bar{D}_{22}$ is given Equation (C-38) and therefore

$$\bar{D}_{22}^t = \sum_i m_i \tilde{R}_{ia} \cdot \tilde{R}_{ia}^t = \sum_i m_i \tilde{R}_{ia} \cdot \tilde{\Phi}_{ia}^{B^t} \dot{\xi}_a^B + \overleftrightarrow{I}_a \cdot \tilde{\omega}^{B^t} - \sum_i m_i \tilde{R}_{ia}^t \vec{R}_{ia} \cdot \vec{\omega}^B \tag{C-60}$$

Now note that the third terms of $\bar{D}_{22}$ and $\bar{D}_{22}^t$ are the negatives of each other, and hence these terms cancel. Consequently

$$\dot{\overleftrightarrow{I}}_a = \dot{\xi}_a^{B^t} \sum_i m_i \tilde{\Phi}_{ia}^B \cdot \tilde{R}_{ia}^t + \sum_i m_i \tilde{R}_{ia} \cdot \tilde{\Phi}_{ia}^{B^t} \dot{\xi}_a^B + \tilde{\omega}^B \cdot \overleftrightarrow{I}_a - \overleftrightarrow{I}_a \cdot \tilde{\omega}^B \tag{C-61}$$

Note that this can also be obtained from Equation (A-58) in the form

$$\dot{\overleftrightarrow{I}}_a = \overset{B}{\overleftrightarrow{I}}_a + \tilde{\omega}^B \cdot \overleftrightarrow{I}_a - \overleftrightarrow{I}_a \cdot \tilde{\omega}^B \tag{C-62}$$

where

$$\overset{B}{\overleftrightarrow{I}}_a = \sum_{j=1}^{n} \dot{\xi}_{aj}^B \frac{\partial \overleftrightarrow{I}_a}{\partial \xi_{aj}^B} = \dot{\xi}_a^{B^t} \nabla_{\xi_a^B} \overleftrightarrow{I}_a \tag{C-63}$$

and where

$$\nabla_{\xi_a^B} \overleftrightarrow{I}_a = \sum_i m_i \left(\nabla_{\xi_a^B} \widetilde{R}_{ia}\right) \cdot \widetilde{R}_{ia}^t + \sum_i m_i \widetilde{R}_{ia} \cdot \left(\nabla_{\xi_a^B} \widetilde{R}_{ia}^t\right)$$

$$= \sum_i m_i \widetilde{\Phi}_{ia}^B \cdot \widetilde{R}_{ia}^t - \sum_i m_i \widetilde{R}_{ia} \cdot \widetilde{\Phi}_{ia}^B \tag{C-64}$$

Evaluation of $\dot{\beta}_a^B$

According to Equation (C-9), $\dot{\beta}_a^B$ is equal to the sum of $\bar{D}_{32}$ and $\bar{D}_{23}^t$. $\bar{D}_{32}$ is given in Equation (C-56). $\bar{D}_{23}$ is given in Equation (C-44); hence $\bar{D}_{23}^t$ is given by

$$\bar{D}_{23}^t = \sum_i m_i \Phi_{ia}^{B^t} \cdot \dot{\widetilde{R}}_{ia}^t = \sum_i m_i \Phi_{ia}^B \cdot \widetilde{\Phi}_{ia}^{B^t} \dot{\xi}_a^B + \vec{\omega}^B \cdot \sum_i m_i \widetilde{R}_{ia} \cdot \widetilde{\Phi}_{ia}^B \tag{C-65}$$

Adding $\bar{D}_{32}$ and $\bar{D}_{23}^t$ yields

$$\dot{\beta}_a^B = \sum_i m_i \Phi_{ia}^B \cdot \widetilde{\Phi}_{ia}^{B^t} \dot{\xi}_a^B + \sum_i m_i \widetilde{R}_{ia} \cdot \left(\nabla_{\xi_a^B} \Phi_{ia}^{B^t}\right)^t \dot{\xi}_a^B + \widetilde{\omega}^a \cdot \beta_a^B \tag{C-66}$$

where we have made use of

$$\widetilde{\beta}_a^B = \sum_i m_i \widetilde{\Phi}_{ia}^B \cdot \widetilde{R}_{ia}^t + \sum_i m_i \widetilde{R}_{ia} \cdot \widetilde{\Phi}_{ia}^B \tag{C-67}$$

plus the fact that $\widetilde{\omega}^a \cdot \beta_a^B = \vec{\omega}^a \cdot \widetilde{\beta}_a^B$.

Note that Equation (C-66) can also be obtained from Equation (B-20) in the form

$$\dot{\beta}_a^B = \overset{B}{\beta}{}_a^B + \widetilde{\omega}^B \cdot \beta_a^B \tag{C-68}$$

where

$$ {}^{B}\beta^{B}_{a} = \sum_{j=1}^{n} \dot{\xi}^{B}_{aj} \frac{\partial \beta^{B}_{a}}{\partial \xi^{B}_{aj}} \tag{C-69} $$

The transpose of ${}^{B}\beta^{B}_{a}$ is actually more easily written, and therefore we write

$$ {}^{B}\beta^{B^t}_{a} = \sum_{j=1}^{n} \dot{\xi}^{B}_{aj} \frac{\partial \beta^{B^t}_{a}}{\partial \xi^{B}_{aj}} = \dot{\xi}^{B^t}_{a} \nabla_{\xi^{B}_{a}} \beta^{B^t}_{a} \tag{C-70} $$

and

$$ \nabla_{\xi^{B}_{a}} \beta^{B^t}_{a} = \sum_i m_i \left(\nabla_{\xi^{B}_{a}} \widetilde{R}_{ia}\right) \cdot \Phi^{B^t}_{ia} + \sum_i m_i \widetilde{R}_{ia} \cdot \nabla_{\xi^{B}_{a}} \Phi^{B^t}_{ia} \tag{C-71} $$

Hence

$$ {}^{B}\beta^{B}_{a} = \left(\nabla_{\xi^{B}_{a}} \beta^{B^t}_{a}\right)^{t} \dot{\xi}^{B}_{a} \tag{C-72} $$

where

$$ \left(\nabla_{\xi^{B}_{a}} \beta^{B^t}_{a}\right)^{t} = \sum_i m_i \Phi^{B}_{ia} \cdot \left(\nabla_{\xi^{B}_{a}} \widetilde{R}_{ia}\right)^{t} + \sum_i m_i \left(\nabla_{\xi^{B}_{a}} \Phi^{B^t}_{ia}\right)^{t} \cdot \widetilde{R}^{t}_{ia} \tag{C-73} $$

Substituting ${}^{B}\beta^{B}_{a}$ from Equations (C-72) and (C-73) into Equation (C-68) yields $\dot{\beta}^{B}_{a}$ as in Equation (C-66).

Evaluation of $\dot{\gamma}_a^B$

According to Equation (C-10), $\dot{\gamma}_a^B$ is equal to the sum $\bar{D}_{33}$ and its transpose. $\bar{D}_{33}$ is given in Equation (C-57), and therefore

$$\bar{D}_{33}^t = \sum_i m_i \Phi_{ia}^B \cdot \dot{\Phi}_{ia}^{B^t} = \sum_i m_i \Phi_{ia}^B \cdot \left[\dot{\xi}_a^{B^t} \nabla_{\xi_a^B} \Phi_{ia}^{B^t} \right] + \sum_i m_i \Phi_{ia}^B \cdot \tilde{\Phi}_{ia}^{B^t} \cdot \vec{\omega}^B$$

(C-74)

Now note that the second terms of $\bar{D}_{33}$ and $\bar{D}_{33}^t$ are the negatives of each other (because of Equation (C-47)), and hence these terms cancel. Consequently

$$\dot{\gamma}_a^B = \sum_i m_i \left[\left(\nabla_{\xi_a^B} \Phi_{ia}^{B^t} \right)^t \dot{\xi}_a^B \right] \cdot \Phi_{ia}^{B^t} + \sum_i m_i \Phi_{ia}^B \cdot \left[\dot{\xi}_a^{B^t} \nabla_{\xi_a^B} \Phi_{ia}^{B^t} \right] \quad \text{(C-75)}$$

We can also get this directly from Equation (C-13) as follows

$$\dot{\gamma}_a^B = \sum_k \dot{\xi}_{ak}^B \frac{\partial \gamma_a^B}{\partial \xi_{ak}^B} = \sum_i m_i \left[\sum_k \dot{\xi}_{ak}^B \left(\frac{\partial \Phi_{ia}^B}{\partial \xi_{ak}^B} \right) \right] \cdot \Phi_{ia}^{B^t} +$$

$$+ \sum_i m_i \Phi_{ia}^B \cdot \left[\sum_k \dot{\xi}_{ak}^B \left(\frac{\partial \Phi_{ia}^{B^t}}{\partial \xi_{ak}^B} \right) \right]$$

(C-76)

We have now obtained explicit expressions for all the elements of $\bar{D}$ (see Equation(4-68)or (C-1)) and $\dot{\bar{\mu}}$ (see Equation(4-74)).